ROMANIAN MATHEMATICAL MAGAZINE

UP.583 Solve for complex numbers:

$$x^8+4x^7+22x^6+52x^5+69x^4+56x^3+28x^2+8x+1=0$$

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Solution by proposer

$$x^8+4x^7+22x^6+52x^5+69x^4+56x^3+28x^2+8x+1=0$$

$$x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1-x^8-4x^7-6x^6-4x^5-x^4+x^8=0$$

$$(x+1)^8-x^4(x^4+4x^3+6x^2+4x+1)+x^8=0$$

$$(x+1)^8-x^4(x+1)^4+x^8=0$$

$$\left(\frac{x+1}{x}\right)^8-\left(\frac{x+1}{x}\right)^4+1=0$$

$$\left(1+\frac{1}{x}\right)^8-\left(1+\frac{1}{x}\right)^4+1=0$$

Denote $\frac{1}{x}=y$

$$(1+y)^8-(1+y)^4+1=0$$

Denote $(1+y)^4=u$

$$u^2-u+1=0$$

$$u_1=\frac{1+i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$u_2=\frac{1-i\sqrt{3}}{2}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$$

$$(1+y)^4=u_1$$

$$(1+y)^4=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$1+y=\sqrt[4]{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\{z_0,z_1,z_2,z_3\}$$

$$z_k=\cos\frac{\frac{\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{\pi}{3}+2k\pi}{4};k\in\overline{0,3}$$

$$z_k=\cos\frac{\pi+6k\pi}{12}+i\sin\frac{\pi+6k\pi}{12};k\in\overline{0,3}$$

$$(1+y)^4=u_2$$

$$1+y=\sqrt[4]{\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}}=\{z'_0,z'_1,z'_2,z'_3\}$$

$$z'_k=\cos\frac{\frac{5\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{5\pi}{3}+2k\pi}{4};k\in\overline{0,3}$$

$$z'_k=\cos\frac{5\pi+6k\pi}{12}+i\sin\frac{5\pi+6k\pi}{12};k\in\overline{0,3}$$

$$y_k=-1+z_k;y'_k=-1+z'_k;k\in\overline{0,3}$$

$$\frac{1}{x_k}=-1+z_k;\frac{1}{x'_k}=-1+z'_k;k\in\overline{0,3}$$

$$x_k=\frac{1}{-1+z_k};x'_k=\frac{1}{-1+z'_k};k\in\overline{0,3}$$

The solutions are:

$$x_k=\frac{1}{-1+\cos\frac{\pi+6k\pi}{12}+i\sin\frac{\pi+6k\pi}{12}};k\in\overline{0,3}$$

$$x'_k=\frac{1}{-1+\cos\frac{5\pi+6k\pi}{12}+i\sin\frac{5\pi+6k\pi}{12}};k\in\overline{0,3}$$