ROMANIAN MATHEMATICAL MAGAZINE

UP.582 Prove that exists $X \in M_{2,3}(\mathbb{R}); Y \in M_{3,2}(\mathbb{R})$ such that:

$$X \cdot Y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; Y \cdot X = \begin{pmatrix} 2 & 6 & 6 \\ 3 & 9 & 9 \\ -3 & -9 & -9 \end{pmatrix}$$

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Solution by proposer

Let be:

$$X = \begin{pmatrix} 1 & a & b \\ 1 & a & b \end{pmatrix}; Y = \begin{pmatrix} 1 & 1 \\ -b & 2b \\ a & -2a \end{pmatrix}; a, b \in \mathbb{R}$$

$$X \cdot Y = \begin{pmatrix} 1 & a & b \\ 1 & a & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -b & 2b \\ a & -2a \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - ab + ab & 1 + 2ab - 2ab \\ 1 - ab + ab & 1 + 2ab - 2ab \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Y \cdot X = \begin{pmatrix} 1 & 1 \\ -b & 2b \\ a & -2a \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 1 & a & b \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot a + 1 \cdot a & 1 \cdot b + 1 \cdot b \\ -b \cdot 1 + 2b \cdot 1 & -b \cdot a + 2b \cdot a & -b \cdot b + 2b \cdot b \\ a \cdot 1 - 2a \cdot 1 & a \cdot a - 2a \cdot a & a \cdot b - 2a \cdot b \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 2a & 2b \\ b & ab & b^2 \\ -a & -a^2 & -ab \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 3 & 9 & 9 \\ -3 & -9 & -9 \end{pmatrix}$$

We take $a = 3; b = 3$ hence:

$$X = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \end{pmatrix}; Y = \begin{pmatrix} 1 & 1 \\ -3 & 6 \\ 3 & -6 \end{pmatrix}$$