ROMANIAN MATHEMATICAL MAGAZINE

UP.581 If $1 \leq a \leq b$ then:

$$24(b-a)^2 \ln\frac{b}{a} + (b^2-a^2)^3 \geq 12(b-a)^2(b^2-a^2) + 24\ln^3\left(\frac{b}{a}\right)$$

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Solution by proposer

The inequality can be written:

$$3(b-a)^2 \ln\frac{b}{a} + \left(\frac{b^2-a^2}{2}\right)^3 \geq 3(b-a)^2\left(\frac{b^2-a^2}{2}\right) + \ln^3\left(\frac{b}{a}\right)$$

$$3\int_a^b dx \cdot \int_a^b dy \cdot \int_a^b \frac{1}{z} dz + \left(\int_a^b x\,dx\right)^3 \geq$$

$$\geq 3\int_a^b dx \int_a^b dy \int_a^b z\,dz + \int_a^b \frac{1}{x} dx \int_a^b \frac{1}{y} dy + \int_a^b \frac{1}{z} dz$$

$$3\int_a^b \int_a^b \int_a^b \frac{1}{x} dx\,dy\,dz + \int_a^b \int_a^b \int_a^b xyz\,dx\,dy\,dz \geq$$

$$\geq 3\int_a^b \int_a^b \int_a^b x\,dx\,dy\,dz + \int_a^b \int_a^b \int_a^b \frac{1}{xyz} dx\,dy\,dz$$

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz\right) dx\,dy\,dz \geq$$

$$\geq \int_a^b \int_a^b \int_a^b \left(x + y + z + \frac{1}{xyz}\right) dx\,dy\,dz$$

It is enough to prove that:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz \geq x + y + z + \frac{1}{xyz} \qquad (1)$$

We will prove by mathematical induction that:

$$P(n): \sum_{i=1}^{n} \frac{1}{x_i} + x_1 x_2 \dots x_n \geq \sum_{i=1}^{n} x_i + \frac{1}{x_1 x_2 \cdot \dots \cdot x_n}$$

For $x_1 = x; x_2 = y; x_3 = z; n = 3$ we obtain (1) by (2).

Suppose $P(n)$ true:

$$P(n+1): \sum_{i=1}^{n+1} \frac{1}{x_i} + x_1 x_2 \dots x_n x_{n+1} \geq \sum_{i=1}^{n+1} x_i + \frac{1}{x_1 x_2 \cdot \dots \cdot x_n x_{n+1}} \qquad \text{(to prove)}$$

$$\sum_{i=1}^{n+1} \frac{1}{x_i} + x_1 x_2 \dots x_n x_{n+1} = \sum_{i=1}^{n} \frac{1}{x_i} + \frac{1}{x_{n+1}} + x_1 x_2 \dots x_n x_{n+1} =$$

$$=\sum_{i=1}^{n}\frac{1}{x_i}+x_1x_2\dots x_n-x_1x_2\cdot\dots\cdot x_n+\frac{1}{x_{n+1}}+x_1x_2\dots x_nx_{n+1}\ge$$

$$\overset{P(n)}{\ge}\sum_{i=1}^{n}x_i+\frac{1}{x_1x_2\dots x_n}-x_1x_2\dots x_n+\frac{1}{x_{n+1}}+x_1x_2\dots x_nx_{n+1}$$

Denote $y=x_1x_2\cdot\dots\cdot x_n$. Remains to prove that:

$$\sum_{i=1}^{n}x_i+\frac{1}{y}-y+\frac{1}{x_{n+1}}+yx_{n+1}\ge\sum_{i=1}^{n+1}x_i+\frac{1}{yx_{n+1}}$$

$$\frac{1}{y}-y+\frac{1}{x_{n+1}}+yx_{n+1}\ge x_{n+1}+\frac{1}{yx_{n+1}}$$

Let's observe that:

$$y\ge 1;x_{n+1}\ge 1;yx_{n+1}\ge 1\text{ or}$$

$$y-1\ge 0;x_{n+1}-1\ge 0;yx_{n+1}-1\ge 0\qquad (3)$$

$$yx_{n+1}-x_{n+1}+\frac{1}{x_{n+1}}-\frac{1}{yx_{n+1}}+\frac{1}{y}-y\ge 0\quad\text{(to prove)}$$

$$x_{n+1}(y-1)+\frac{1}{x_{n+1}}\left(1-\frac{1}{y}\right)-\frac{y^2-1}{y}\ge 0$$

$$x_{n+1}(y-1)+\frac{1}{x_{n+1}}\cdot\frac{y-1}{y}-\frac{(y-1)(y+1)}{y}\ge 0$$

$$(y-1)\left(x_{n+1}+\frac{1}{yx_{n+1}}-\frac{y+1}{y}\right)\ge 0$$

$$(y-1)\left(x_{n+1}-1+\frac{1}{yx_{n+1}}-\frac{1}{y}\right)\ge 0$$

$$(y-1)\left(x_{n+1}-1-\frac{x_{n+1}-1}{yx_{n+1}}\right)\ge 0$$

$$(y-1)(x_{n+1}-1)\left(1-\frac{1}{yx_{n+1}}\right)\ge 0$$

$$(y-1)(x_{n+1}-1)(yx_{n+1}-1)\ge 0\quad\text{(true by (3))}$$

Equality holds for $a=b$.