UP.580 If $f:[a,b]\to[-1,\infty)$; $a,b\in\mathbb{R}$; $a\le b$; f continuous, then:

$$\left(\int_a^b(1+f(x))dx\right)^3\ge(b-a)^3+3(b-a)^2\int_a^b f(x)\,dx$$

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Solution by proposer

First we recall Bernoulli's generalized inequality:

If $n\in\mathbb{N}^*; x_1,x_2,\dots,x_n\in[-1,\infty); x_i\cdot x_j\ge 0; (\forall)i,j\in\overline{1,n}$ then:

$$(1+x_1)(1+x_2)\cdot\ldots\cdot(1+x_n)\ge 1+x_1+x_2+\cdots+x_n \qquad (1)$$

For $n=2$ we must prove that:

$$(1+x_1)(1+x_2)\ge 1+x_1+x_2$$

$$1+x_1+x_2+x_1x_2\ge 1+x_1+x_2$$

$$x_1x_2\ge 0 \quad \text{(True)}$$

$P(n):\prod_{i=1}^n(1+x_i)\ge 1+\sum_{i=1}^n x_i$ (suppose true)

$P(n+1):\prod_{i=1}^{n+1}(1+x_i)\ge 1+\sum_{i=1}^{n+1} x_i$ (to prove)

$$\prod_{i=1}^{n+1}(1+x_i)=(1+x_{n+1})\cdot\prod_{i=1}^{n}(1+x_i)\overset{P(n)}{\ge}$$

$$\ge(1+x_{n+1})\cdot\left(1+\sum_{i=1}^n x_i\right)=$$

$$=1+\sum_{i=1}^n x_i+x_{n+1}+x_{n+1}\cdot\sum_{i=1}^n x_i\ge 1+\sum_{i=1}^{n+1}x_i$$

$$\sum_{i=1}^{n+1}x_i+\sum_{i=1}^{n}x_i\cdot x_{n+1}\ge\sum_{i=1}^{n+1}x_i$$

$$\sum_{i=1}^n x_i\cdot x_{n+1}\ge 0 \quad \text{(true)}$$

$$P(n)\to P(n+1)$$

We take in (1):

$$x_1=f(x); x_2=f(y); x_3=f(z); x_1,x_2,x_3\in[-1,\infty)$$

$$(1+f(x))(1+f(y))(1+f(z))\ge 1+f(x)+f(y)+f(z)$$

By integrating:

$$\int_a^b \int_a^b \int_a^b \big(1+f(x)\big)\big(1+f(y)\big)\big(1+f(z)\big)dxdydz \geq$$

$$\geq \int_a^b \int_a^b \int_a^b \big(1+f(x)+f(y)+f(z)\big)\,dxdydz$$

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)\left(\int_a^b f(z)\,dz\right) \geq \int_a^b \int_a^b \int_a^b dx\,dy\,dz +$$

$$+\int_a^b \int_a^b \int_a^b f(x)\,dx\,dy\,dz + \int_a^b \int_a^b \int_a^b f(y)\,dx\,dy\,dz + \int_a^b \int_a^b \int_a^b f(z)\,dx\,dy\,dz$$

$$\left(\int_a^b f(x)\,dx\right)^3 \geq (b-a)^3 + 3\int_a^b f(x)\,dx \cdot \int_a^b \int_a^b dy\,dz$$

$$\left(\int_a^b f(x)\,dx\right)^3 \geq (b-a)^3 + 3(b-a)^2 \int_a^b f(x)\,dx$$

Equality holds for $a=b$.