ROMANIAN MATHEMATICAL MAGAZINE

UP.578 Find a closed form:

$$\int_{-1}^{1} \frac{1}{\sqrt[5]{(1-x)^2(1+x)^3}}\,dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I = \int_{-1}^{1} \frac{1}{\sqrt[5]{(1-x)^2(1+x)^3}}\,dx$$

In this integral we make the variable change: $1 + x = 2t$.

We obtain:

$$I = \int_0^1 t^{-\frac{3}{5}}(1-t)^{-\frac{2}{5}}\,dt.$$

We use Euler's Beta function:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \text{ where } p, q > 0$$

We set conditions:

$$p - 1 = -\frac{3}{5};\, q - 1 = -\frac{2}{5}$$

Result:

$$p = \frac{2}{5};\, q = \frac{3}{5}$$

So:

$$I = B\left(\frac{2}{5}, \frac{3}{5}\right) = \frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{\Gamma(1)} = \Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{50}\right)$$

where $\Gamma(a)$ is Euler's Gamma function.

We use the complement formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

We have:

$$I=\frac{\pi}{\sin\left(\frac{2\pi}{5}\right)}$$

The relationship is known:

$$\sin\left(\frac{2\pi}{5}\right)=\frac{1}{4}\sqrt{10+2\sqrt{5}}$$

We obtain:

$$I=\frac{4}{\sqrt{10+2\sqrt{5}}}\pi$$

After some elementary calculations we arrive at:

$$I=\sqrt{2-\frac{2}{\sqrt{5}}}\pi$$

We obtained the value of the integral required in the problem statement.