ROMANIAN MATHEMATICAL MAGAZINE

UP.578 Prove the equality:

$$\int_0^\infty \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx = \frac{1}{8}\pi^2\sqrt{5-\frac{7}{\sqrt{2}}}$$

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Solution by proposer

Let us denote:

$$I = \int_0^\infty \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx;$$

$$A = \int_0^1 \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx = \int_0^1 \frac{(1-x)x^4\sqrt{x}\ln x}{1-x^8}dx$$

$$B = \int_1^\infty \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx = \int_1^\infty \frac{(1-x)x^4\sqrt{x}\ln x}{1-x^8}dx$$

We consider the integral A.

We have:

$$A = \int_0^1 \frac{x^{\frac{9}{2}}\ln x}{1-x^8}dx - \int_0^1 \frac{x^{\frac{11}{2}}\ln x}{1-x^8}dx$$

We will calculate:

$$A_1 = \int_0^1 \frac{x^{\frac{9}{2}}\ln x}{1-x^8}dx$$

We make the variable change: $x^8 = y; x = y^{\frac{1}{8}}$

We obtain:

$$A_1 = \frac{1}{64}\int_0^1 \frac{y^{-\frac{5}{16}}\ln y}{1-y}dy$$

We will use the following known relationship:

$\int_0^1 \frac{z^a \ln z}{1-z}dz = -\psi_1(a+1), a \in R \setminus \{-1,-2,\dots\},$ where $\psi_1(x)$ is the trigamma function

We obtain:

$$A_1=-\frac{1}{64}\psi_1\left(\frac{11}{16}\right)$$

We will calculate:

$$A_2=\int_0^1\frac{x^{\frac{11}{2}}\ln x}{1-x^8}dx$$

Proceeding similarly, we obtain:

$$A_2=-\frac{1}{64}\psi_1\left(\frac{13}{16}\right)$$

We obtained the value of the integral A:

$$A=A_1-A_2=\frac{1}{64}\left[-\psi_1\left(\frac{11}{16}\right)+\psi_1\left(\frac{13}{16}\right)\right]$$

We conside the integral B. We make the variable change: $x=\frac{1}{t}$.

We obtain:

$$B=\int_0^1\frac{(t-1)t^{\frac{1}{2}}\ln t}{1-t^8}dt$$

By proceeding similarly to the integral A, we obtain:

$$B=\frac{1}{64}\left[\psi_1\left(\frac{3}{16}\right)-\psi_1\left(\frac{5}{16}\right)\right]$$

Result:

$$I=A+B$$

$$I=\frac{1}{64}\left[-\psi_1\left(\frac{11}{16}\right)+\psi_1\left(\frac{13}{16}\right)+\psi_1\left(\frac{3}{16}\right)-\psi_1\left(\frac{5}{16}\right)\right]$$

We use the reflection formula:

$$\psi_1+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{3}{16}\right)+\psi_1\left(\frac{13}{16}\right)=\frac{\pi^2}{\sin^2\frac{3\pi}{16}};\ \psi_1\left(\frac{5}{16}\right)+\psi_1\left(\frac{11}{16}\right)=\frac{\pi^2}{\sin^2\frac{5\pi}{16}};$$

Result:

$$I=\frac{\pi^2}{64}\left(\frac{1}{\sin^2\frac{3\pi}{16}}-\frac{1}{\sin^2\frac{5\pi}{16}}\right)$$

We have:

$$\frac{1}{\sin^2\frac{3\pi}{16}}-\frac{1}{\sin^2\frac{5\pi}{16}}=32\sin^3\frac{\pi}{8}$$

We propose to the reader to prove this equality.

Result:

$$I=\frac{1}{2}\pi^2\sin^3\frac{\pi}{8}$$

We have:

$$\sin\frac{\pi}{8}=\frac{1}{2}\sqrt{2-\sqrt{2}}$$

We propose to the reader to prove this equality.

Result:

$$I=\frac{1}{8}\pi^2\sqrt{5-\frac{7}{\sqrt{2}}}$$

Thus, the problem is solved.