ROMANIAN MATHEMATICAL MAGAZINE

UP.577 Find a closed form:

$$\int_0^1 \frac{x^2 \ln x}{x^3 + x\sqrt{x} + 1} dx$$

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Solution by proposer

Let us denote:

$$I = \int_0^1 \frac{x^2 \ln x}{x^3 + x\sqrt{x} + 1} dx$$

In this integral we make the variable change: $x = z^{\frac{2}{3}}$.

We obtain:

$$I = \frac{4}{9}\int_0^1 \frac{z \ln z}{z^2 + z + 1} dz.$$

We have, successively:

$$I = \frac{4}{9}\int_0^1 \frac{(1-z) z \ln z}{1 - z^3} dz$$

$$I = \frac{4}{9}\left(\int_0^1 \frac{z \ln z}{1 - z^3} dz - \int_0^1 \frac{z^2 \ln z}{1 - z^3} dz\right)$$

$$I = \frac{4}{9}\left(\int_0^1 \sum_{n=0}^{\infty} z^{3n+1} \ln z\, dz - \int_0^1 \sum_{n=0}^{\infty} z^{3n+2} \ln z\, dz\right)$$

$$I = \frac{4}{9}\sum_{n}\left(\int_0^1 z^{3n+1} \ln z\, dz - \int_0^1 z^{3n+2} \ln z\, dz\right).$$

We will use the following relationship:

$$\int_0^1 x^a \ln x\, dx = -\frac{1}{(a+1)^2}, \text{ where } a \in \mathbb{R}, a \geq 0.$$

We obtain:

$$I = \frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{1}{(3n+3)^2} - \frac{1}{(3n+2)^2}\right].$$

Or:

$$I=\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{(n+1)^2}-\frac{\frac{1}{9}}{\left(n+\frac{2}{3}\right)^2}\right].$$

We now use the following relationship:

$$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\psi_1(x)$ is the trigamma function.

We have:

$$I=\frac{4}{81}\left[\psi_1(1)-\psi_1\left(\frac{2}{3}\right)\right].$$

The following special value of triagamma function is known:

$$\psi_1(1)=\frac{\pi^2}{6}$$

We obtained the value of the integral required in the problem statement:

$$I=\frac{4}{81}\left[\frac{\pi^2}{6}-\psi_1\left(\frac{2}{3}\right)\right].$$