ROMANIAN MATHEMATICAL MAGAZINE

UP.576 Find a closed form:

$$\int_0^{\infty}\frac{\arctan(x)}{\sqrt[5]{x^{10}+1}}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote: $A=\int_0^{\infty}\frac{\arctan(x)}{\sqrt[5]{x^{10}+1}}dx$.

We also consider the integral: $B=\int_0^{\infty}\frac{\operatorname{arccot}(x)}{\sqrt[5]{x^{10}+1}}dx$.

We have:

$$A+B=\int_0^{\infty}\frac{\arctan(x)+\operatorname{arccot}(x)}{\sqrt[5]{x^{10}+1}}dx=\frac{\pi}{2}\int_0^{\infty}\frac{1}{\sqrt[5]{x^{10}+1}}dx$$

We are going to calculate the integral:

$$C=\int_0^{\infty}\frac{1}{\sqrt[5]{x^{10}+1}}dx.$$

We use the following definition of Euler's Beta function:

$$B(p,q)=\int_0^{\infty}\frac{y^{p-1}}{(1+y)^{p+q}}dy.$$

In the C integral we make the following variable change: $x^{10}=y$.

We obtain:

$$C=\frac{1}{10}\int_0^{\infty}\frac{y^{-\frac{9}{10}}}{(1+y)^{\frac{1}{5}}}dy.$$

For $p=\frac{1}{10}$ and $q=\frac{1}{10}$ we can write:

$$C=\frac{1}{10}B\left(\frac{1}{10},\frac{1}{10}\right).$$

We use the known relationship:

$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where $\Gamma(a)$ is the Euler's Gamma function.

We can write:

$$C = \frac{1}{10} \frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

So we can write:

$$A + B = \frac{\pi}{20} \frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

In the B integral we make the variable change $x = \frac{1}{t}$ and we immediately obtain

$$B = A.$$

So we have:

$$2A = \frac{\pi}{20} \frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

We obtain the value of the integral required in the problem statement:

$$A = \frac{\pi}{40} \frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

Thus, the problem is solved.