UP.575 Calculate:

$$2\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dy\,dx$$

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Solution 1 by proposer

Firstly, we can express

$$\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dx\,dy=\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx$$

Secondly, it is straightforward to confirm that

$$\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1-y}{x^2+y^2}+\frac{\arctan\left(\frac{y}{x}\right)}{x}\right)=\frac{y(y-1)}{(x^2+y^2)^2}$$

Therefore

$$2\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dy\,dx=\int_0^1\left[\frac{1-y}{x^2+y^2}+\frac{\arctan\left(\frac{y}{x}\right)}{x}\right]_1^\infty dx=$$

$$=\int_0^1\left(\frac{\frac{\pi}{2}}{x}-\frac{\arctan\left(\frac{1}{x}\right)}{x}\right)dx=\int_0^1\frac{\arctan(x)}{x}\,dx=\int_0^1\sum_{n=0}^\infty\frac{(-1)^n x^{2n}}{2n+1}\,dx=$$

$$=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^2}=G$$

Solution 2 by proposer

Firstly, we can express

$$\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dx\,dy=\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx$$

Secondly, substituting y by $\frac{1}{y}$, we obtain

$$\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx=\int_0^1\int_0^1\frac{\frac{1}{y}\left(\frac{1}{y}-1\right)}{\left(x^2+\frac{1}{y^2}\right)^2}\left(\frac{1}{y^2}dy\right)dx=\int_0^1\int_0^1\frac{(1-y)}{(1+x^2y^2)^2}\,dy\,dx=$$

$$= \int_0^1 \int_0^1 (1-y)\left(\sum_{n=1}^{\infty} (-1)^{n-1} n x^{2(n-1)} y^{2(n-1)}\right) dy\, dx =$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n \left(\int_0^1 x^{2n-2} dx\right)^2 - \sum_{n}^{\infty} (-1)^{n-1} n \left(\int_0^1 x^{2n-2}\, dx\right)\left(\int_0^1 y^{2n-1}\, dy\right) =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(2n-1)(2n)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(2n-1)^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(2n-1)^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)}{(2n-1)^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2}.$$

Therefore:

$$2 \int_0^1 \int_0^{\infty} \frac{y(y+1)}{(x^2 + (y+1)^2)^2}\, dy\, dx = G.$$