UP.574 If $0 \le a \le b \le 1$ then:

$$\int_a^b \int_a^b \int_a^b \frac{dx\,dy\,dz}{\sqrt{1+xyz}} \ge (b-a)^2 \int_a^b \frac{dx}{\sqrt{1+x^3}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be $f:[0,1] \to \mathbb{R}; f(x) = \frac{1}{\sqrt{1+e^{3x}}}$

$$f'(x) = \frac{-3e^{3x}}{2(1+e^{3x})\sqrt{1+e^{3x}}};$$

$$f''(x) = \frac{-3e^{3x}(2e^{3x}-1)}{4(1+e^{3x})^2\sqrt{1+e^{3x}}} < 0; x \in [0,1]$$

$$x \in [0,1] \Rightarrow 0 \le x \le 1 \Rightarrow 0 \le 3x \le 3 \Rightarrow 1 \le e^{3x} \le e^3$$

$$2 \le 2e^{3x} \le 2e^3 \Rightarrow 1 \le 2e^{3x} - 1 \le 2e^3 - 1$$

$$2e^{3x} - 1 \ge 1 \Rightarrow 2e^{3x} - 1 > 0 \Rightarrow -3e^{3x}(2e^{3x}-1) < 0$$

$f''(x) < 0 \Rightarrow f$ concave. By Jensen's inequality:

$$f(a) + f(b) + f(c) \le 3f\left(\frac{a+b+c}{3}\right)$$

$$\frac{1}{\sqrt{1+e^{3a}}} + \frac{1}{\sqrt{1+e^{3b}}} + \frac{1}{\sqrt{1+e^{3c}}} \le \frac{3}{\sqrt{1+e^{a+b+c}}}$$

Denote: $x = e^a; y = e^b; z = e^c \Rightarrow$

$$\frac{1}{\sqrt{1+x^3}} + \frac{1}{\sqrt{1+y^3}} + \frac{1}{\sqrt{1+z^3}} \le \frac{3}{\sqrt{1+xyz}}$$

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{\sqrt{1+x^3}} + \frac{1}{\sqrt{1+y^3}} + \frac{1}{\sqrt{1+z^3}}\right) dx\,dy\,dz \le 3\int_a^b \int_a^b \int_a^b \frac{dx\,dy\,dz}{\sqrt{1+xyz}}$$

$$3(b-a)^2 \int_a^b \frac{dx}{\sqrt{1+x^3}} \le 3\int_a^b \int_a^b \int_a^b \frac{dx\,dy\,dz}{\sqrt{1+xyz}}$$

Equality holds for $a = b$.