UP.573 Let be $\lambda > 1$ fixed. Find:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k+1}{\lambda^k}$$

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Solution by proposer

We have the sum of the geometric progression:

$$1+x+x^2+\cdots+x^n=\frac{x^{n+1}-1}{x-1}, x\neq 1.$$

We derive the equality above:

$$1+2x+3x^2+\cdots+nx^{n-1}=\left(\frac{x^{n+1}-1}{x-1}\right)'=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}.$$

Multiplying by $x, x\neq 0$, the above equality:

$$x+2x^2+3x^3+\cdots+nx^n=\frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}.$$

Replacing $x=\frac{1}{\lambda}$ we obtain:

$$\sum_{k=0}^{n}\frac{k+1}{\lambda^k}=\frac{1}{\lambda^k}=\frac{1}{\lambda}+\frac{1}{\lambda^2}+\frac{3}{\lambda^3}+\cdots+\frac{n}{\lambda^n}=\frac{\frac{n}{\lambda^{n+2}}-\frac{n+1}{\lambda^{n+1}}+\frac{1}{\lambda}}{\left(\frac{1}{\lambda}-1\right)^2}.$$

Because $\frac{n}{\lambda^{n+2}}\to 0$ and $\frac{n+1}{\lambda^{n+1}}\to 0$ it follows that $\frac{\frac{n}{\lambda^{n+2}}-\frac{n+1}{\lambda^{n+1}}+\frac{1}{\lambda}}{\left(\frac{1}{\lambda}-1\right)^2}\to\frac{\lambda}{(\lambda-1)^2}$

Then:

$$\sum_{k=0}^{n}\frac{1}{\lambda^k}=1+\frac{1}{\lambda}+\frac{1}{\lambda^2}+\cdots+\frac{1}{\lambda^n}=\frac{1-\frac{1}{\lambda^{n+1}}}{1-\frac{1}{\lambda}}\to\frac{\lambda}{\lambda-1}$$

We obtain $\lim\limits_{n\to\infty}\sum_{k=0}^{n}\frac{k}{\lambda^k}=\frac{\lambda}{(\lambda-1)^2}$ and $\lim\limits_{n\to\infty}\sum_{k=0}^{n}\frac{1}{10^k}=\frac{\lambda}{\lambda-1}$ so

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k+1}{\lambda^k}=\left(\frac{\lambda}{\lambda-1}\right)^2$$