UP.572 Prove that $25/17$ is the largest positive value of the constant k such that:

$$\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}\ge\frac{4}{1+k}$$

for any nonnegative real numbers a,b,c,d with at most one of them larger than 1 and $ab+ac+ad+bc+bd+cd=6.$

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Solution by proposer

Without loss of generality, assume that $a,b,c\in[0,1]$ and $d\ge 1$. For $b=c=1$, the inequality becomes

$$\frac{1}{a^2+k}+\frac{1}{d^2+k}\ge\frac{2}{1+k},$$

under the constraint $ad+4S=5$, where $S=\frac{a+d}{2}$. From $5=ad+4S\le S^2+4S$ and $5=ad+4S\ge 4S$, we get $1\le S\le\frac{5}{4}$. Write the inequality as follows:

$$\frac{4S^2-2ad+2k}{(ad-k)^2+4kS^2}\ge\frac{2}{1+k},$$

$$\frac{4S^2-2(5-4S)+2k}{(5-k-4S)^2+4kS^2}\ge\frac{2}{1+k},$$

$$(S-1)[15-3k-(k+7)S]\ge 0$$

Choosing $S=\frac{5}{4}$, we get the necessary condition $k\le\frac{25}{17}$. So, we only need to prove the original inequality for $k=\frac{25}{17}$.

If c,d are fixed, then the expression

$$F=\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}$$

has the minimum value when

$$E(a,b)=\frac{1}{a^2+k}+\frac{1}{b^2+k}$$

has the minimum value. Denoting

$$x=a,y=b,A=c+d,B=6-cd,$$

we have $A > 0, B \geq 0, x, y \in [0,1]$ and $xy + A(x+y) = B$. For $B = 0$, we have $x = y = 0$ (hence $a = b = 0$), while for $B > 0$, by Lemma below, it follows that $E(a,b)$ (hence F, too) has the minimum value for $\min\{a,b\} = 0$ or $a = b$. By extending this result to any two of a, b, c, it suffices to prove the original inequality for $a = b = c$, for $a = 0$ and $b = c$, and for $a = b = 0$.

Case 1: $a = b = c$. We need to show that

$$\frac{3}{c^2+k} + \frac{1}{d^2+k} \geq \frac{4}{1+k}$$

for $0 \leq c \leq 1 \leq d$ such that $c^2 + cd = 2$. The inequality is equivalent to

$$(1-c^2)^2(3-k-c^2) \geq 0.$$

It is true because $3 - k - c^2 \geq 2 - k > 0$.

Case 2: $a = 0$ and $b = c$. We need to show that:

$$\frac{1}{k} + \frac{2}{c^2+k} + \frac{1}{d^2+k} \geq \frac{4}{1+k}$$

for $0 \leq c \leq 1 \leq d$ such that $c^2 + 2cd = 6$. Write the inequality as follows:

$$\frac{2}{c^2+k} - \frac{2}{1+k} \geq \frac{2}{1+k} - \frac{1}{k} - \frac{1}{d^2+k}$$

$$\frac{2(1-c^2)}{(1+k)(c^2+k)} \geq \frac{(k-1)(1-c^2)(36-c^2)}{k(1+k)[c^4 - 4(3-k)c^2 + 36]}$$

It is true if

$$\frac{2}{c^2+k} \geq \frac{(k-1)(36-c^2)}{k[c^4 - 4(3-k)c^2 + 36]},$$

i.e.

$$\frac{25}{c^2+k} \geq \frac{4(36-c^2)}{c^4 - 4(3-k)c^2 + 36}$$

It is true if

$$\frac{24}{1+k} \geq \frac{4 \cdot 36}{0 - 4(3-k) + 36},$$

i.e.

$$\frac{2}{1+k} \geq \frac{3}{6+k}, \quad 9 \geq k.$$

Case 3: $a = b = 0$. We need to show that

$$\frac{2}{k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}\ge\frac{4}{1+k}$$

for $0<c\le 1\le d$ such that $cd=6$. It is true if

$$\frac{2}{k}+\frac{1}{1+k}+0\ge\frac{4}{1+k},$$

i.e. $k\le 2$.

The proof is completed. For $k=\frac{25}{17}$, the equality occurs when $a=b=c=d=1$, and also for $a=0, b=c=1$ and $d=\frac{5}{2}$ (or any cyclic permutation).

Lemma. Let A and B be positive real constants, and let $x,y\in[0,1]$ such that

$$xy+A(x+y)=B.$$

If $k>1$, then the expression

$$E=\frac{1}{x^2+k}+\frac{1}{y^2+k}$$

has the minimum value for $\min\{x,y\}=0$ or $x=y$.

Proof.

Let $s=x+y$ and $p=xy$. We need to show that if

$$0\le 4p\le s^2$$

and

$$p+As=B,$$

then the expression

$$E=\frac{s^2-2p+2k}{ks^2+(p-k)^2}$$

has the minimum value for $p=0$ (when $\min\{x,y\}=0$) or $4p=s^2$ (when $x=y$). From

$$B=p+As\ge p+2A\sqrt{p},$$

we get

$$p\le p_1=\left(\sqrt{A^2+B}-A\right)^2,$$

with equality for $4p=s^2$. Since

$$kE=1+\frac{k^2-p^2}{ks^2+(p-k)^2}=1+F(p),$$

where

$$F(p)=\frac{A^2(k^2-p^2)}{(A^2+k)p^2-2k(A^2+B)p+k^2A^2+kB^2},$$

it suffices to show that $F(p)$ has the minimum value m when $p=0$ or $p=p_1$. Write the inequality $F(p)\ge m$ as $F_1(p)\ge 0$, where

$$F_1(p)=-[(m+1)A^2+km]p^2+2km(A^2+B)p-(m-1)k^2A^2-kmB^2.$$

Since $F_1(p)$ is concave on $[0,p_1]$, it has the minimum value 0 when $p=0$ or $p=p_1$.