ROMANIAN MATHEMATICAL MAGAZINE

UP.571 Prove that is the least value of the constant $k > 2$ such that:

$$\left(a^k+b^k+c^k\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$

for any positive real numbers a, b, c with at most one of them less than 1 and

$$a^5+b^5+c^5=3$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

By choosing $b=1, c\in(0,1]$ and $a=\sqrt[5]{2-c^5}$, the constraints are satisfied, and the inequality is equivalent to $E(c)\geq 0$, where

$$E(c)=\left(a^k+c^k+1\right)\left(\frac{1}{a}+\frac{1}{c}+1\right)-9.$$

Note that $c=1$ implies $a=1$. From $a^5+c^5=2$, we get

$$a'(c)=\frac{-c^4}{a^4},\quad a'(1)=-1,$$

$$a''(c)=\frac{4c^4a'}{a^5}-\frac{4c^3}{a^4},\quad a''(1)=-8,$$

and

$$E'(c)=k\left(a^{k-1}a'+c^{k-1}\right)\left(\frac{1}{a}+\frac{1}{c}+1\right)+\left(a^k+c^k+1\right)\left(\frac{-a'}{a^2}-\frac{1}{c^2}\right),$$

$$E''(c)=k\left[(k-1)a^{k-2}(a')^2+a^{k-1}a''+(k-1)c^{k-2}\right]\left(\frac{1}{a}+\frac{1}{c}+1\right)$$

$$+2k\left(a^{k-1}a'+c^{k-1}\right)\left(\frac{-a'}{a^2}-\frac{1}{c^2}\right)+\left(a^k+c^k+1\right)\left(\frac{-a''}{a^2}+\frac{2(a')^2}{a^3}+\frac{2}{c^3}\right).$$

We have $E(1)=0, E'(1)=0$ and

$$E''(1)=3k(k-1-8+k-1)+0+3(8+2+2)=6(k-2)(k-3).$$

Since $E(1)=E'(1)=0$, the condition $E''(1)\geq 0$ is necessary to have $E(c)\geq 0$ for $c\in(0,1]$. This condition implies $k\geq 3$. To show that 3 is the least value of the constant k. we need to show that $F\geq 0$, where

$$F=\left(a^3+b^3+c^3\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-9.$$

Let

$$a \geq b \geq 1 \geq c.$$

For fixed a, assume that c and F are functions of b. By differentiating the equality constraint, we get

$$b^4 + c^4 a' = 0, \quad c' = -b^4 c^{-4},$$

therefore

$$F'(b) = 3(b^2 + c^2 c')\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + (a^3 + b^3 + c^3)\left(\frac{-1}{b^2} + \frac{-c'}{c^2}\right)$$

$$= 3(b^2 - b^4 c^{-2})\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + (a^3 + b^3 + c^3)\left(\frac{-1}{b^2} + b^4 c^{-6}\right).$$

We will show that $F'(b) \geq 0$. Denoting $x = \frac{b}{c}$, we have

$$F'(b) = -3c^2(x^4 - x^2)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + b^{-2}(a^3 + b^3 + c^3)(x^6 - 1).$$

Since $x \geq 1$ and $a \geq b$, we have

$$F'(b) = -3c^2(x^4 - x^2)\left(\frac{2}{b} + \frac{1}{c}\right) + b^{-2}(2b^3 + c^3)(x^6 - 1),$$

hence

$$\frac{F'(b)}{c} \geq -3(x^4 - x^2)\left(\frac{2}{x} + 1\right) + \left(2x + \frac{1}{x^2}\right)(x^6 - 1)$$

$$= \frac{(x^2 - 1)\left(2x^7 + 2x^5 - 2x^4 - 4x^3 + x^2 + 1\right)}{x^2}$$

$$= \frac{(x^2 - 1)(x - 1)\left(2x^6 + 2x^5 + 4x^4 + 2x^3 - 2x^2 - x - 1\right)}{x^2} \geq 0.$$

From $F'(b) \geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. Thus, it suffices to prove the desired inequality for $b = 1$. We need to show that

$$(a^3 + c^3 + 1)\left(\frac{1}{a} + \frac{1}{c} + 1\right) \geq 9$$

for $a \geq 1 \geq c > 0$ such that $a^5 + c^5 = 2$. Let $S = \frac{a+c}{2} > \frac{1}{2}$ and $p = ac$. From the known inequality $a^5 + c^5 \geq 2S^5$, we get $S \leq 1$, and from

$$2 = a^5 + c^5 = (a^2 + c^2)(a^3 + c^3) - a^2c^2(a + c) = (4S^2 - 2p)(8S^3 - 6Sp) - 2Sp^2,$$

we obtain

$$5Sp^2 = 20S^3p - 16S^5 + 1,\ \ p = 2S^2 - \sqrt{\frac{4S^5+1}{5S}}.$$

We claim that

$$p \leq \frac{5S^2-1}{4},$$

that is

$$\sqrt{\frac{4S^5+1}{5S}} \geq \frac{3S^2+1}{4}.$$

By squaring, the inequality becomes

$$16(4S^5+1) \geq 5S(3S^2+1)^2,\quad 19S^5 - 30S^3 - 5s + 16 \geq 0,$$

$$(S-1)^2(19S^3 + 38S^2 + 27S + 16) \geq 0$$

Write now the desired inequality as follows:

$$(8S^3 - 6Sp + 1)\left(\frac{2S}{p} + 1\right) \geq 9,\ \ 3Sp^2 + 2(2 + 3S^2 - 2S^3)p - S(8S^3+1) \leq 0,$$

$$\frac{3(20S^3p - 16S^5 + 1)}{5} + 2(2 + 3S^2 - 2S^3)p - S(8S^3+1) \leq 0,$$

$$48S^5 + 40S^4 + 5S - 3 \geq 10(4S^3 + 3S^2 + 2)p.$$

Since $p \leq \frac{5S^2-1}{4}$, we have

$$48S^5 + 40S^4 + 5S - 3 - 10(4S^3 + 3S^2 + 2)p \geq$$

$$\geq 48S^5 + 40S^4 + 5S - 3 - \frac{5(4S^3 + 3S^2 + 2)(5S^2-1)}{2}$$

$$= \frac{4 + 10S - 35S^2 + 20S^3 + 5S^4 - 4S^5}{2} = \frac{(1-S)^2(4 + 18S - 3S^2 - 4S^3)}{2} \geq 0.$$

For $k = 3$ the equality occurs when $a = b = c = 1$.