ROMANIAN MATHEMATICAL MAGAZINE

SP.585 Let $f:[n-1,n]\to[n,n+1]$ be a continuous function such that

$$\int_{n-1}^{n}\left(1+xf'(x)\right)dx\le nf(n)-(n-1)f(n-1),$$

then prove:

$$\int_{n-1}^{n}\frac{dx}{f(x)}\le\frac{2}{n+1},n\in\mathbb{N}^*$$

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Solution by proposer

$$\int_{n-1}^{n}\left(1+xf'(x)\right)dx\le nf(n)-(n-1)f(n-1)\Rightarrow$$

$$nf(n)-(n-1)f(n-1)-\int_{n-1}^{n}xf'(x)\,dx\ge 1$$

But, using Integration by Parts method, we have:

$$\int_{n-1}^{n}xf'(x)=xf(x)|_{n-1}^{n}-\int_{n-1}^{n}f(x)\,dx$$

$$\int_{n-1}^{n}xf'(x)\,dx=nf(n)-(n-1)f(n-1)-\int_{n-1}^{n}f(x)\,dx$$

$$\int_{n-1}^{n}f(x)\,dx=nf(n)-(n-1)f(n-1)-\int_{n-1}^{n}xf'(x)\,dx\ge 1$$

For all $x\in[n-1,n]$ and $n\in\mathbb{N}^*$ we have $\frac{(f(x)-n)(f(x)-n-1)}{f(x)}\le 0$ or

$$f(x)-(2n+1)+\frac{n(n+1)}{f(x)}\le 0\Leftrightarrow\frac{1}{f(x)}\le\frac{2n+1}{n(n+1)}-\frac{1}{n(n+1)}f(x)\Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx\le\frac{2n+1}{n(n+1)}\int_{n-1}^{n}dx-\frac{1}{n(n+1)}\int_{n-1}^{n}f(x)\,dx\Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx\le\frac{2n+1}{n(n+1)}\left(n(n-1)\right)-\frac{1}{n(n+1)}\Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx\le\frac{2n+1}{n(n+1)}-\frac{1}{n(n+1)}\Leftrightarrow\int_{n-1}^{n}\frac{1}{f(x)}dx\le\frac{2}{n+1}$$