ROMANIAN MATHEMATICAL MAGAZINE

SP.584 Let be the sequence $(x_n)n \geq 1$ defined by

$$x_1 = 1, x_{n+2} = 3x_{n+1} - x_n, \forall n \in \mathbb{N}. \text{ Find:}$$

$$L = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{x_{2k+1}}{x_k + x_{k+1}}}$$

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Solution by proposers

The sequence $(x_n)_{n\geq 0}$ verify a recurrence relationship by second order, then

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right], \forall n \in \mathbb{N}.$$

We show that: $x_1 + x_3 + x_5 + \cdots + x_{2n-1} = x_n^2, \forall n \in \mathbb{N}^*$. We have:

$$x_1 + x_3 + x_5 + \cdots + x_{2n-1} = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\left(\frac{3+\sqrt{5}}{2}\right)^{2k-1} - \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\left(\frac{3-\sqrt{5}}{2}\right)^{2k-1}$$

Using the formula of a geometrical progression, it follows:

$$x_1 + x_3 + x_5 + \cdots + x_{2n-1} = \frac{1}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{2n} + \left(\frac{3-\sqrt{5}}{2}\right)^{2n} - 5\right] = x_n^2$$

$$\begin{cases} x_1 + x_3 + x_5 + \cdots + x_{2k-1} = x_k^2 \\ x_1 + x_3 + x_5 + \cdots + x_{2k+1} = x_{k+1}^2 \end{cases} \overset{(-)}{\Longrightarrow}$$

$$x_{2k+1} = x_{k+1}^2 - x_k^2 \text{ or } \frac{x_{2k+1}}{x_k + x_{k+1}} = x_{k+1} - x_k$$

$$\sum_{k=0}^{n} \frac{x_{2k+1}}{x_k + x_{k+1}} = \sum_{k=0}^{n}(x_{k+1} - x_k) = x_{n+1}$$

Therefore:

$$\lim_{n\to\infty} \sqrt[n]{x_{n+1}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3+\sqrt{5}}{2}\right)^{n+1}\right)} =$$

$$= \lim_{n\to\infty}\left(\frac{3+\sqrt{5}}{2}\right)\cdot\sqrt[n]{\frac{1}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{2} - \left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)(3-\sqrt{5})\right]} = \frac{3+\sqrt{5}}{2}$$