SP.580 If $a \in \mathbb{R}; m, n \geq 1$ then:

$$(2+\cos a)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$

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Solution by proposer

Lemma 1: If $x > 1; p \geq 1$ then:

$$(x-1)^p \leq x^p - 1 \quad (1)$$

Proof.

Let be $f:(1,\infty) \to \mathbb{R}; f(x) = (x-1)^p - x^p + 1$

$$f'(x) = p(x-1)^{p-1} - px^{p-1} = p((x-1)^{p-1} - x^{p-1}) < 0$$

because $x - 1 < x; (\forall) x > 1$

f decreasing on $(1,\infty)$

$$\sup_{x>1} f(x) = \lim_{\substack{x \to 1 \\ x>1}} f(x) = \lim_{\substack{x \to 1 \\ x>1}} ((x-1)^p - x^p + 1) =$$

$$= (1-1)^p - 1^p + 1 = -1 + 1 = 0$$

$$f(x) \leq 0; (\forall) x > 1 \Rightarrow (x-1)^p - x^p + 1 \leq 0$$

$$(x-1)^p \leq x^p - 1$$

Equality holds for $p = 1$.

Lemma 2: If $x > 1; m, n \geq 1$ then:

$$(x-1)^{m+n} + x^m + x^n \leq x^{m+n} + 1 \quad (2)$$

By (1):

$$(x-1)^m \leq x^m - 1 \quad (3)$$

$$(x-1)^n \leq x^n - 1 \quad (4)$$

By multiplying (3); (4):

$$(x-1)^m \cdot (x-1)^n \leq (x^m - 1)(x^n - 1)$$

$$(x-1)^{m+n} \leq x^{m+n} - x^m - x^n + 1$$

$$(x-1)^{m+n} + x^m + x^n \leq x^{m+n} + 1$$

Back to the problem:

We take $x = 3 + \cos a > 1$ in (2):

$$(3+\cos a - 1)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$

$$(2+\cos a)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$

Equality holds for $m = n = 1$:

$$(2+\cos a)^2 + (3+\cos a)^1 + (3+\cos a)^1 = (3+\cos a)^2 + 1$$

$$4 + 4\cos a + \cos^2 a + 6 + 2\cos a = 9 + \cos^2 a + 6\cos a + 1$$

$$\cos^2 a + 6\cos a + 10 = \cos^2 a + 6\cos a + 10$$