ROMANIAN MATHEMATICAL MAGAZINE

SP.579 If $f:[a,b]\to\mathbb{R};\ 0<a\le b;\ f$ – continuous then:

$$\frac{b^{4047}-a^{4047}}{4047}+\int_a^b f^2(x^{2024})\,dx\ge\frac{1}{1012}\int_{a^{2024}}^{b^{2024}} f(x)\,dx$$

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Solution by proposer

$$\left(x^{2023}-f(x^{2024})\right)^2\ge 0\Rightarrow\int_a^b\left(x^{2023}-f(x^{2024})\right)^2dx\ge 0$$

$$\int_a^b x^{4046}\,dx+\int_a^b f^2(x^{2024})\,dx-2\int_a^b x^{2023}f(x^{2024})dx\ge 0 \qquad (1)$$

For the integral $\int_a^b x^{2023}f(x^{2024})dx$ denote:

$$y=x^{2024}\Rightarrow dy=2024x^{2023}dx$$

If $x=a\Rightarrow y=a^{2024}$

If $x=b\Rightarrow y=b^{2024}$

$$\int_a^b x^{2023}f(x^{2024})dx=\int_{a^{2024}}^{b^{2024}} f(y)\cdot\frac{1}{2024}dy=\frac{1}{2024}\int_{a^{2024}}^{b^{2024}} f(x)\,dx \qquad (2)$$

Replacing (2) in (1):

$$\int_a^b x^{4047}\,dx+\int_a^b f^2(x^{2024})\,dx-2\cdot\frac{1}{2024}\int_{a^{2024}}^{b^{2024}} f(x)\,dx\ge 0$$

$$\frac{b^{4048}-a^{4048}}{4048}+\int_a^b f^2(x^{2024})\,dx\ge\frac{1}{1012}\int_{a^{2024}}^{b^{2024}} f(x)\,dx$$

Equality holds for $a=b$.