

ROMANIAN MATHEMATICAL MAGAZINE

SP.578 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\log_{\sin x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) + \log_{\cos x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) \geq 2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\begin{aligned} & \sqrt{\frac{\log_{\sin x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) + \log_{\cos x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right)}{2}} \geq \\ & \stackrel{QM-HM}{\geq} \frac{2}{\frac{1}{\log_{\sin x} \left(\frac{\sin 2x}{\sin x + \cos x} \right)} + \frac{1}{\log_{\cos x} \left(\frac{\sin 2x}{\sin x + \cos x} \right)}} = \\ & = \frac{2}{\log_{\left(\frac{\sin 2x}{\sin x + \cos x} \right)} \sin x + \frac{1}{\log_{\cos x} \left(\frac{\sin 2x}{\sin x + \cos x} \right)}} = \\ & = \frac{2}{\log_{\left(\frac{\sin 2x}{\sin x + \cos x} \right)} \sin x + \log_{\left(\frac{\sin 2x}{\sin x + \cos x} \right)} \cos x} = \\ & = \frac{2}{\log_{\left(\frac{\sin 2x}{\sin x + \cos x} \right)} (\sin x + \cos x)} = 2 \log_{\sin x + \cos x} \left(\frac{\sin 2x}{\sin x + \cos x} \right) \geq 1 \\ & \Leftrightarrow \log_{\sin x \cos x} \left(\frac{\sin 2x}{\sin x + \cos x} \right)^2 \geq \log_{\sin x \cos x} (\sin x \cos x) \\ & \Leftrightarrow \left(\frac{\sin 2x}{\sin x + \cos x} \right)^2 \leq \sin x \cos x \\ & \Leftrightarrow 4 \sin^2 x \cos^2 x \leq \sin x \cos x (\sin x + \cos x)^2 \\ & \Leftrightarrow 4 \sin x \cos x \leq (\sin x + \cos x)^2 \Leftrightarrow 4 \sin x \cos x \leq \sin^2 x + \cos^2 x + 2 \sin x \cos x \\ & \Leftrightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x \geq 0 \Leftrightarrow (\sin x - \cos x)^2 \geq 0 \end{aligned}$$

Equality holds for $x = \frac{\pi}{4} \Leftrightarrow \sin x = \cos x$

$$\sqrt{\frac{\log_{\sin x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) + \log_{\cos x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right)}{2}} \geq 1$$

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$$\frac{\log_{\sin x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) + \log_{\cos x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right)}{2} \geq 1$$

$$\log_{\sin x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) + \log_{\cos x}^2 \left(\frac{\sin 2x}{\sin x + \cos x} \right) \geq 2$$