ROMANIAN MATHEMATICAL MAGAZINE

SP.577 Find all $n \in \mathbb{N}^*$ such that:

$$\int_0^1 (\sin x)^{2n-2} \cdot (\cos x)^{2n} dx \geq \frac{1}{4^{1011}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\frac{1}{4^{1011}} \leq \int_0^1 (\sin x)^{2n-2} \cdot (\cos x)^{2n} dx =$$

$$= \int_0^1 (\sin^2 x)^{n-1} \cdot (\cos^2 x)^n dx = \int_0^1 \cos^2 x \cdot (\sin^2 x \cdot \cos^2 x)^{n-1} dx =$$

$$= \int_0^1 \cos^2 x \cdot \left(\sin^2 x\,(1 - \sin^2 x)\right)^{n-1} dx \overset{AM-GM}{\leq}$$

$$\leq \int_0^1 \cos^2 x \cdot \left(\left(\frac{\sin^2 x + 1 - \sin^2 x}{2}\right)^2\right)^{n-1} dx =$$

$$= \int_0^1 \cos^2 x \cdot \frac{1}{2^{2(n-1)}} dx = \frac{1}{4^{n-1}} \int_0^1 \cos^2 x\, dx < \frac{1}{4^{n-1}} \cdot \int_0^1 dx = \frac{1}{4^{n-1}}$$

$$\frac{1}{4^{1011}} < \frac{1}{4^{n-1}} \Rightarrow 4^{n-1} < 4^{1011} \Rightarrow n - 1 < 1011$$

$$\Rightarrow n < 1012 \Rightarrow n \in \{1, 2, 3, \dots, 1011\}$$