ROMANIAN MATHEMATICAL MAGAZINE

SP.576 If $a, b, c \geq 1$, then:

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}}-\frac{a+b+c}{3} \geq \sqrt[3]{\frac{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}{3}}-\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

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Solution by proposer

Without loss of generality, we may assume that $c \geq b \geq a \geq 1$.

We write the inequality in the form $E(a,b,c) \geq 0$, where:

$$E(a,b,c)=\sqrt[3]{9(a^3+b^3+c^3)}-(a+b+c)-\sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

We shall prove that: $E(a,b,c) \geq E(a,x,x) \geq 0$, where:

$$E(a,x,x)=\sqrt[3]{9(a^3+2x^3)}-(a+2x)-\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}+\left(\frac{1}{a}+\frac{2}{x}\right)$$

and $x=\sqrt{bc} \geq a$;

a. We will prove the inequality: $E(a,b,c) \geq E(a,x,x)$.

The inequality can be written as follows:

$A-B \geq C-D$, where: $A=\sqrt[3]{9(a^3+b^3+c^3)}-\sqrt[3]{9(a^3+2x^3)}$;

$$A=\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(a^3+b^3+c^3)}\right)^2+\sqrt[3]{9(a^3+b^3+c^3)}\sqrt[3]{9(a^3+2x^3)}+\left(\sqrt[3]{9(a^3+2x^3)}\right)^2}$$

$$A \geq \frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}$$

$$B=a+b+c-(a+2x)=b+c-2x=\left(\sqrt{b}-\sqrt{c}\right)^2$$

$$C=\sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}-\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}$$

$$C=\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^6\left(\left(\sqrt[3]{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}\right)^2+\sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}+\left(\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\right)^2\right)}$$

$$C\le\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^6\left(\left(\sqrt[3]{9\left(\frac{1}{x^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\right)^2+\sqrt[3]{9\left(\frac{1}{x^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\frac{3}{x}+\frac{9}{x^2}\right)}$$

$$C\le\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}$$

$$D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\left(\frac{1}{a}+\frac{2}{x}\right)=\frac{1}{b}+\frac{1}{c}-\frac{2}{x}=\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

To prove the inequality $E(a,b,c)\ge E(a,x,x)$ it is enough to prove that:

$$\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}-\left(\sqrt{b}-\sqrt{c}\right)^2\ge$$

$$\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}-\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

Or:

$$\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{9\left(b+c+\sqrt{bc}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}-1\right)\ge$$

$$\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{9\left(b+c+\sqrt{bc}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}-\frac{1}{x^2}\right)$$

For $b=c$ this relationship it is true (case of equality). For $b\ne c$ we will show that:

$$\frac{9\left(b+c+\sqrt{bc}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}\left(1-\frac{1}{x^2}\right)\ge1-\frac{1}{x^2}$$

$$9\left(b+c+\sqrt{bc}\right)^2\ge\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2$$

$$9\left(b+c+\sqrt{bc}\right)^2 \geq \left(\sqrt[3]{9\left(bc\sqrt{bc}+b^3+c^3\right)}\right)^2 + \sqrt[3]{9\left(bc\sqrt{bc}+b^3+c^3\right)}3\sqrt{bc}+9bc$$

Let us denote: $y=\frac{b}{c}; 0<y\leq 1$

The previous inequality is written in the equivalent form:

$$9\left(y+1+\sqrt{y}\right)^2 \geq \left(\sqrt[3]{9\left(y\sqrt{y}+y^3+1\right)}\right)^2 + \sqrt[3]{9\left(y\sqrt{y}+y^3+1\right)}3\sqrt{y}+9y$$

But, we have for any $y, 0<y\leq 1$: $\quad 9\left(y\sqrt{y}+y^3+1\right) \leq \left(\frac{9}{10}y+\frac{21}{10}\right)^3$

Indeed, we calculate:

$$\left(\frac{9}{10}y+\frac{21}{10}\right)^3 - 9\left(y\sqrt{y}+y^3+1\right) =$$

$$=\frac{9}{1000}\left(1-\sqrt{y}\right)\left(1352y+919y^2+29\sqrt{y}+352y\sqrt{y}+919y^2\sqrt{y}+29\right)\geq 0$$

Returning to the main inequality, it suffices to prove:

$$9\left(y+1+\sqrt{y}\right)^2 \geq \left(\frac{9}{10}y+\frac{21}{10}\right)^2 + \left(\frac{9}{10}y+\frac{21}{10}\right)3\sqrt{y}+9y$$

$$100\left(y+1+\sqrt{y}\right)^2-(3y+7)^2-10(3y+7)\sqrt{y}-100y=$$

$$=158y+91y^2+130\sqrt{y}+170y\sqrt{y}+51\geq 0$$

Then, the inequality $E(a,b,c)\geq E(a,x,x)$ is proved.

b. We will prove the inequality: $E(a,x,x)\geq 0$. We have to show that:

$$\sqrt[3]{9(a^3+2x^3)}-(a+2x)\geq \sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}-\left(\frac{1}{a}+\frac{2}{x}\right)$$

The inequality is successively written in the following equivalent forms:

$$\frac{9(a^3+2x^3)-(a+2x)^3}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2+\sqrt[3]{9(a^3+2x^3)}(a+2x)+(a+2x)^2}\geq$$

$$\geq \frac{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)-\left(\frac{1}{a}+\frac{2}{x}\right)^3}{\left(\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\right)^2+\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\left(\frac{1}{a}+\frac{2}{x}\right)+\left(\frac{1}{a}+\frac{2}{x}\right)^2}$$

$$\frac{2(4a+5x)(a-x)^2}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2+\sqrt[3]{9(a^3+2x^3)}(a+2x)+(a+2x)^2}\geq$$

$$\geq\frac{\frac{1}{a^3x^3}2(5a+4x)(a-x)^2}{\frac{1}{a^2x^2}\left(\left(\sqrt[3]{9(x^3+2a^3)}\right)^2+\sqrt[3]{9(x^3+2a^3)}(x+2a)+(x+2a)^2\right)}$$

$$\frac{2(4a+5x)(a-x)^2}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2+\sqrt[3]{9(a^3+2x^3)}(a+2x)+(a+2x)^2}\geq$$

$$\geq\frac{2(5a+4x)(a-x)^2}{ax\left(\sqrt[3]{9(x^3+2a^3)}\right)^2+ax\sqrt[3]{9(x^3+2a^3)}(x+2a)+ax(x+2a)^2}$$

The last inequality holds because:

a. $4a+5x\geq 5a+4x$**, equivalent to:** $x\geq a$.

b. $ax\left(\sqrt[3]{9(x^3+2a^3)}\right)^2\geq\left(\sqrt[3]{9(a^3+2x^3)}\right)^2$**, equivalent to:**

$$a^3(x^9-a^3)+4x^6(a^6-1)+4a^3x^3(a^6-1)\geq 0$$

c. $ax\sqrt[3]{9(x^3+2a^3)}(x+2a)\geq\sqrt[3]{9(a^3+2x^3)}(a+2x)$**, results from inequalities b and d**

d. $ax(x+2a)^2\geq(a+2x)^2$**, equivalent to:**

$$a(x^3-a)+4x^2(a^2-1)+4ax(a^2-1)\geq 0$$

Thus, the inequality $E(a,x,x)\geq 0$ **is proved.**

So, the inequality in the statement: $E(a,b,c)\geq 0$ **is also proved.**