ROMANIAN MATHEMATICAL MAGAZINE

SP.573 Let $\lambda \geq \frac{1}{2}$ fixed. If $a, b, c > 0, a + b + c \leq 4$ then find the maximum value of

$$P = \frac{\sqrt{abc}}{(a+b)(b+c)} - \frac{\lambda}{c(a+b)}$$

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Solution by proposer

Lemma 1: If $a, b, c > 0$ then:

$$\frac{\sqrt{abc}}{(a+b)(b+c)} \leq \frac{1}{2\sqrt{a+b+c}}$$

Proof:

Using means inequality we obtain:

$$(a+b)(b+c) = b^2 + ab + bc + ca = b(a+b+c) + ac \overset{AGM}{\geq} 2\sqrt{b(a+b+c)\cdot ac} =$$

$$= 2\sqrt{abc(a+b+c)}, \text{ with equality for } b(a+b+c) = ac.$$

Lemma 2: If $a, b, c > 0$ then:

$$\frac{1}{c(a+b)} \geq \frac{4}{(a+b+c)^2}.$$

Proof:

With means inequality we obtain:

$$c + (a+b) \overset{AGM}{\geq} 2\sqrt{c(a+b)}, \text{ with equality for } c = a+b, \text{so } a+b+c \geq 2\sqrt{c(a+b)} \Rightarrow$$

$$\Rightarrow (a+b+c)^2 \geq 4c(a+b) \Rightarrow c(a+b) \leq \frac{(a+b+c)^2}{4} \Rightarrow \frac{1}{c(a+b)} \geq \frac{4}{(a+b+c)^2}.$$

Let's get back to the main problem. Using the above Lemmas, we obtain:

$$P = \frac{\sqrt{abc}}{(a+b)(a+c)} - \frac{\lambda}{c(a+b)} \leq \frac{1}{2\sqrt{a+b+c}} - \frac{4\lambda}{(a+b+c)^2}$$

Denoting $\sqrt{a+b+c} = t$ we have $0 < t \leq 2$ and $P \leq \frac{1}{2t} - \frac{4\lambda}{t^4}$.

Considering the function $f: (0,2] \to \mathbb{R}, f(t) = \frac{1}{2t} - \frac{4\lambda}{t^4}$.

We have $f'(t) = \frac{32\lambda - t^3}{t^5} > 0, 0 < t \le 2, \lambda \ge \frac{1}{2}$. It follows that the function f is strictly increasing on $(0, 2]$, so $f(t) \le f(2) = \frac{1}{4} - \frac{\lambda}{9}$

It follows $P \le \frac{1}{2t} - \frac{4\lambda}{t^4} = f(t) \le f(2) = \frac{1-\lambda}{4}$.

From $P \le \frac{1}{4} - \frac{\lambda}{9}$, with equality for $a + b + c = 4, b(a + b + c) = ac, c = a + b$, namely for $(a, b, c) = \left(\frac{4}{3}, \frac{2}{3}, 2\right)$, we deduce that $\max P = \frac{1-\lambda}{4}$ and the maximum is attained for $(a, b, c) = \left(\frac{4}{3}, \frac{2}{3}, 2\right)$.