SP.572 Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \geq 1$, and let

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Prove that:

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

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Solution by proposer

Since $F(a,b,c) \geq 0$ and $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) \geq 0$ (by the AM-GM inequality), it suffices to prove the homogeneous inequality

$$F(a,b,c) \geq (a^4bc)^{\frac{1}{3}} \cdot F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \geq 1$. Thus, we need to show that

$$(bc)^{\frac{1}{3}} - \frac{3bc}{b+c+bc} \geq (bc)^{\frac{1}{3}}\left[\frac{1}{(bc)^{\frac{1}{3}}} - \frac{3}{1+b+c}\right].$$

Denote

$$s = \frac{b+c}{2}, \quad p = \sqrt{bc},$$

with $s \geq p \geq 1$. The desired inequality is equivalent to

$$p^{\frac{2}{3}} - \frac{3p^2}{2s+p^2} \geq 1 - \frac{3p^{\frac{2}{3}}}{2s+1},$$

$$p^{\frac{2}{3}} - 1 \geq 3p^{\frac{2}{3}}\left(\frac{p^{\frac{4}{3}}}{2s+p^2} - \frac{1}{2s+1}\right),$$

$$\frac{p^{\frac{2}{3}}}{3p^{\frac{2}{3}}} \geq \frac{2s\left(p^{\frac{4}{3}}-1\right)-p^{\frac{4}{3}}\left(p^{\frac{2}{3}}-1\right)}{(2s+p^2)(2s+1)}.$$

It is true if

$$\frac{1}{3p^{\frac{2}{3}}} \geq \frac{2s\left(p^{\frac{2}{3}}+1\right)-p^{\frac{4}{3}}}{(2s+p^2)(2s+1)},$$

i.e.

$$4s^2-2As+4p^2 \geq 0, \quad A=3p^{\frac{4}{3}}+3p^{\frac{2}{3}}-p^2-1.$$

For the nontrivial case $A \geq 0$, since

$$4(4s^2-2As+4p^2)=(4s-A)^2+16p^2-A^2 \geq 16p^2-A^2=(4p-A)(4p+A),$$

it suffices to show that $4p-A \geq 0$, which is equivalent to

$$p^2-3p^{\frac{4}{3}}+4p-3p^{\frac{2}{3}}+1 \geq 0.$$

Denoting $p=x^3$, we need to show that

$$x^6-3x^4+4x^3-3x^2+1 \geq 0,$$

that is

$$(x-1)^2(x^4+2x^3+2x+1) \geq 0.$$

The proof is completed. The equality occurs for $a=b=c \geq 1$.

Remark. The inequality $F(a,b,c) \leq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ is true in the particular case $a,b,c \geq 1$

(which involves $a^4bc \geq 1$).