SP.571 For given $n \geq 3$, prove that 3 is the largest positive value of the constant k such that:

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n \geq k(a_1+a_2+\cdots+a_n-n)$$

for any $a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n > 0$ with

$$a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1=n.$$

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Solution by proposer

Choosing $a_2=\cdots=a_{n-1}=1$, then inequality becomes $\frac{1}{a_1}+\frac{1}{a_n}-2 \geq k(a_1+a_n-2)$, where $a_1 \geq 1 \geq a_n > 0$ such that $a_1a_n+a_1+a_n=3$. Let $p=a_1a_n$. From $3=a_1a_n+a_1+a_n \geq p+2\sqrt{p}$, we get $p \in (0,1]$. Write the inequality as follows:

$$\frac{3-p}{p}-2 \geq k(1-p), \quad (1-p)(3-kp) \geq 0.$$

It is true if and only if $3-kp \geq 0$ for $p \in (0,1)$. From the necessary condition $\lim_{p\to 1}(3-kp) \geq 0$, we get $k \leq 3$. To show that 3 is the largest positive value of k, we need to prove the inequality

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+2n \geq 3(a_1+a_2+\cdots+a_n).$$

By the AM-HM inequality, we have $\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}} \geq \frac{n-2}{S}$, where $S=\frac{a_2+\cdots+a_{n-1}}{n-2} \geq 1$. So, it suffices to show that $E \geq 0$, where

$$E=\frac{1}{a_1}+\frac{1}{a_n}+\frac{n-2}{S}+2n-3[a_1+a_n+(n-2)S].$$

By Lemma below, we have $(n-3)S^2+(a_1+a_n)S+a_1a_n < n$. Since the expression E decreases when a_1 increases, we may increase a_1 to have

$$(n-3)S^2+(a_1+a_n)S+a_1a_n=n.$$

Denoting $x=\frac{a_1+a_n}{2}$, we need to show that

$$\frac{2x}{n-(n-3)S^2-2Sx}+\frac{n-2}{S}+2n-3[2x+(n-2)S] \geq 0$$

for $(n-3)S^2+2Sx+a_1a_2=n$. From $(S-a_1)(S-a_n) \leq 0$, we obtain:

$$2Sx \geq a_1a_n+S^2=n-2Sx-(n-4)S^2$$

therefore

$$4Sx \geq n-(n-4)S^2.$$

For fixed S, the desired inequality is equivalent to $F(x) \geq 0$, where

$$F(x)=12S^2x^2+[6(2n-5)S^2-4nS-8n+6]Sx+$$
$$+[n-(n-3)S^2][n-2+2nS-3(n-2)S^2]$$

Since

$$F'(x)=24S^2x+6(2n-5)S^3-4nS^2-(8n-6)S \geq$$
$$\geq 6S[n-(n-4)S^2]+6(2n-5)S^3-4nS^2-(8n-6)S$$
$$=6(n-1)S^3-4nS^2-(2n-6)S \geq 6(n-1)S^2-4nS^2-(2n-6)S=$$
$$=n(n-3)S(S-1) \geq 0,$$

$F(x)$ is increasing, hence

$$F(x) \geq F\left(\frac{n-(n-4)S^2}{4S}\right)=\frac{n[3(n-2)S^4-4(n-2)S^3-2nS^2+4nS-n-2]}{4}$$
$$=\frac{n(S-1)^2[3(n-2)S^2+2(n-2)S-n-2]}{4} \geq 0.$$

The proof is completed. For $k=3$, the equality occurs when $a_1=a_2=\cdots=a_n=1$.

Lemma:

Let $n \geq 3$. If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ such that $a_1a_2+a_2a_3+\cdots+a_na_1=n$, then

$$(n-3)S^2+(a_1+a_n)S+a_1a_n \leq n,$$

where $S=\frac{a_2+\cdots+a_{n-1}}{n-2}$.

Proof:

For $n=3$, the inequality is an equality. For $n \geq 4$, we write the desired inequality in the homogeneous form:

$$(n-3)S^2+(a_1+a_n)S+a_1a_n \leq a_1a_2+a_2a_3+\cdots+a_na_1,$$

which is equivalent to

$$(n-3)S^2+a_1(S-a_2)+a_n(S-a_{n-1}) \leq a_2a_3+\cdots+a_{n-2}a_{n-1}.$$

Since $S-a_2 \leq 0$ and $S-a_{n-1} \geq 0$, it suffices to show that

$$(n-3)S^2+a_2(S-a_2)+a_{n-1}(S-a_{n-1}) \leq a_2a_3+\cdots+a_{n-2}a_{n-1},$$

which can be rewritten as

$$a_2a_3+\cdots+a_{n-2}a_{n-1} \geq (n-3)S^2+(a_2+a_{n-1})S-a_2^2-a_{n-1}^2.$$

Since the sequence $a_2, a_3, \dots, a_{n-2}$ and $a_3, a_4, \dots, a_{n-1}$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3+\cdots+a_{n-2}a_{n-1}) \geq (a_2+\cdots+a_{n-2})(a_3+\cdots+a_{n-1}) =$$
$$=\big((n-2)S-a_{n-1}\big)\big((n-2)S-a_2\big).$$

Thus, it suffices to show that

$$\frac{\big((n-2)S-a_{n-1}\big)\big((n-2)S-a)2\ \big)}{n-3} \geq (n-3)S^2+(a_2+a_{n-1})S-a_2^2-a_{n-1}^2,$$

which is equivalent to

$$(2n-5)S^2-(2n-5)(a_2+a_{n-1})S+(n-3)\big(a_2^2+a_{n-1}^2\big)+a_2a_{n-1} \geq 0,$$
$$(2n-5)(2S-a_2-a_{n-1})^2+(2n-7)(a_2-a_{n-1})^2 \geq 0.$$

Clearly, the last inequality is true.