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JP.585 In $\Delta ABC, AA', BB', CC'$ are internal bisectors which intersect the circumcircle of triangle in the points A'', B'', C''. Prove that:

$$6\left(\left(\frac{R}{r}\right)^2-2\right)\le\frac{AA''}{A'A''}+\frac{BB''}{B'B''}+\frac{CC''}{C'C''}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

$$\frac{AA''}{A'A''}=\frac{AA'+A'AA''}{A'A''}=\frac{AA'}{A'A''}+1=\frac{AA''^2}{AA'\cdot A'A''}+1=\frac{w_a^2}{BA'\cdot A'C}+1 \quad (1)$$

But, from Law of bisector: $BA'\cdot A'C=\frac{a^2bc}{(b+c)^2}$ (2)

From (1) and (2), it follows:

$$\frac{AA''}{A'A''}=\frac{4b^2c^2}{(b+c)^2}\cdot\cos\frac{A}{2}\cdot\frac{(b+c)^2}{a^2bc}+1=\frac{4bc}{a^2}\cdot\cos^2\frac{A}{2}=$$

$$=\frac{4s(s-a)}{a^2}+1=\frac{(b+c+a)(b+c-a)}{a^2}+1=$$

$$=\frac{(b+c)^2-a^2}{a^2}+1=\frac{(b+c)^2}{a^2} \text{ (and analogs)}$$

We must show that: $6\left(\left(\frac{R}{r}\right)^2-2\right)\le\sum_{cyc}\frac{(b+c)^2}{a^2}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{b+c}{a}\right)^2=\frac{1}{3}\left(\frac{s^2+r^2-2Rr}{2Rr}\right)^2\ge$$

$$\overset{Gerretsen}{\ge}\frac{1}{3}\left(\frac{14Rr-4r^2}{2Rr}\right)^2=\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\overset{CBS}{\le}\sum_{cyc}\frac{2(b^2+c^2)}{a^2}=2\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)$$

But, $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\frac{R^2}{r^2}-2$. Finally, we get:

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\le 2\left(\frac{3R^2}{r^2}-6\right)=6\left(\frac{R^2}{r^2}-2\right)$$