ROMANIAN MATHEMATICAL MAGAZINE

JP.579 Solve for real numbers:

$$\begin{cases} \dfrac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x-y)+yz(y-z)+zx(z-x)}=55 \\ x^3+y^3+z^3=99 \\ \dfrac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}=\dfrac{55}{9} \end{cases}$$

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Solution by proposer

Denote: $S_1=x+y+z; S_2=xy+yz+zx; S_3=xyz.$

$$\frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x-y)+yz(y-z)+zx(z-x)}=55$$

$$\frac{(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)}{(x-y)(y-z)(z-x)}=55$$

$$x^2+y^2+z^2+xy+yz+zx=55,\qquad S_1^2-2S_2+S_2=55\Rightarrow S_1^2+S_2=55$$

$$\frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}=\frac{55}{9}$$

$$\frac{x^2+y^2+z^2+xy+yz+zx}{x+y+z}=\frac{55}{9}$$

$$\frac{S_1^2-S_2}{S_1}=\frac{55}{9}\Rightarrow\frac{55}{S_1}=\frac{55}{9}\Rightarrow S_1=9$$

$$S_1^2-S_2=55\Rightarrow 9^2-S_2=55\Rightarrow 81-S_2=55\Rightarrow S_2=26$$

$$x^3+y^3+z^3=99,\qquad S_1\left(S_1^2-3S_2\right)+3S_3=99$$

$$9(81-3\cdot 26)+3S_3=99\Rightarrow 9(81-78)+3S_3=99$$

$$3S_3=99-27\Rightarrow S_3=33-9\Rightarrow S_3=24$$

Let be the equation with roots x,y,z**:**

$$u^3-S_1u^2+S_2u-S_3=0$$

$$u^3-9u^2+26u-25=0$$

$$(u-2)(u-3)(u-4)=0\Rightarrow u_1=2;u_2=3;u_3=4$$

Solutions: $(x,y,z)=(2,3,4)$ **and permutations.**