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JP.578 If $x,y,z>0$ and $n\in\mathbb{N}, n\ge 2$, in ΔABC holds:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}\ge\frac{\sqrt{3}}{2r}(6r^2)^n$$

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Solution by proposer

Lemma:

If $x,y,z>0$ and $n\in\mathbb{N}, n\ge 2$, in ΔABC holds:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}\ge\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n$$

Proof:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}=\sum\frac{\left(\frac{x}{y+z}a^2\right)^n}{a}\overset{Holder}{\ge}\frac{\left(\sum\frac{x}{y+z}a^2\right)^n}{3^{n-2}\sum a}\overset{Tsintsifas}{\ge}\frac{\left(2\sqrt{3}F\right)^n}{3^{n-2}\cdot 2p}=$$

$$=\frac{9}{2p}\left(\frac{2\sqrt{3}F}{3}\right)^n=\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n$$

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum\frac{x^{2n}a^{2n+1}}{(y+z)^{2n}}\overset{Lemma}{\ge}\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n=\frac{9}{2p}\cdot\frac{2^n p^n r^n}{\left(\sqrt{3}\right)^n}=\frac{9}{2}\cdot\frac{2^n p^{n-1} r^n}{\left(\sqrt{3}\right)^n}\ge$$

$$\overset{Mitrinovic}{\ge}\frac{9}{2}\cdot\frac{2^n\left(3\sqrt{3}r\right)^{n-1}r^n}{\left(\sqrt{3}\right)^n}=$$

$$=\frac{3^2}{2}\cdot\frac{2^n(3r)^{n-1}r^n}{\sqrt{3}}=\frac{3\sqrt{3}}{2}\cdot\frac{2^n\cdot 3^n r^{n-1}r^n}{3}=\frac{\sqrt{3}}{2}\cdot 6^n\cdot r^{2n-1}=\frac{\sqrt{3}}{2r}(6r^2)^n=RHS$$

Equality holds if and only if the triangle is equilateral. We have used above:

Lemma Tsintsifas:

If $x,y,z>0$ then in ΔABC holds:

$$\sum\frac{x}{y+z}a^2\ge 2\sqrt{3}F$$

Proof:

$$\sum\frac{x}{y+z}a^2=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\frac{x+y+z}{y+z}a^2-\sum a^2\overset{Bergstrom}{\ge}$$

$$\ge(x+y+z)\frac{(\sum a)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{(2p)^2}{2(x+y+z)}-2(p^2-r^2-4Rr)=$$

$$= 2p^2 - 2(p^2 - r^2 - 4Rr) = 2(r^2 + 4Rr) = 2r(4R + r).$$

We have used above the known identities in the triangle:

$$\sum a = 2p \text{ and } \sum a^2 = 2(p^2 - r^2 - 4Rr)$$

It remains to prove that:

$$2r(4R + r) \geq 2\sqrt{3}S \Leftrightarrow r(4R + r) \geq \sqrt{3}rp \Leftrightarrow 4R + r \geq p\sqrt{3},$$

which is Doucet's inequality.

Equality holds if and only if the triangle is equilateral.