ROMANIAN MATHEMATICAL MAGAZINE

JP.577 If $x, y, z > 0, xyz = 1$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{x^{2n+1}}{x+y+1} \geq 1$$

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Solution by proposer

Lemma: If $x, y, z > 0, xyz = 1$ and $k \in \mathbb{N}$ then:

$$\sum x^{k+1} \geq \sum x^k$$

Proof:

Using Chebyshev's inequality we obtain:

$$\sum x^{k+1} = \sum x \cdot x^k \overset{Chebyshev}{\geq} \frac{1}{3} \sum x \sum x^k \overset{AGM}{\geq} \sqrt[3]{xyz} \cdot \sum x^k = \sqrt[3]{1} \cdot \sum x^k = \sum x^k$$

with equality for $x = y = z = 1$. Let's get back to the main problem.

Using Lemma we obtain:

$$LHS = \sum \frac{x^{2n+1}}{x+y+1} = \sum \frac{x^{2n+2}}{x^2+xy+x} \overset{CS}{\geq} \frac{(\sum x^{n+1})^2}{\sum(x^2+xy+x)} = \frac{(\sum x^{n+1})^2}{\sum x^2 + \sum xy + \sum x} \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} \frac{(\sum x^{n+1})^2}{\sum x^{n+1} + \sum x^{n+1} + \sum x^{n+1}} = \frac{(\sum x^{n+1})^2}{3 \sum x^{n+1}} = \frac{\sum x^{n+1}}{3} \overset{(1)}{\geq} 1 = RHD$$

where (1) $\Leftrightarrow \sum x^{n+1} \geq \sum x^2, \sum x^{n+1} \geq \sum x^2 \geq \sum xy, \sum x^{n+1} \geq \sum x, \sum x^{n+1} \geq 3$, see Lemma. Equality holds if and only if $x = y = z = 1$.

Note.

For $n = 2$ we obtain the Proposed problem by Kostantinos Geronikolas in Mathematical Inequalities 2/2022: If $x, y, z > 0, xyz = 1$ then:

$$\sum \frac{x^5}{x+y+1} \geq 1$$

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