ROMANIAN MATHEMATICAL MAGAZINE

JP.576 If $a,b,c>0, a+b+c=3$ and $0\le\lambda\le 3$ then:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+\frac{\lambda abc}{a^2b+b^2c+c^2a+abc}\ge\frac{\lambda+6}{4}$$

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Solution by proposer

$$LHS=\sum\frac{a^2}{bc}\overset{CS}{\ge}\frac{(\sum a)^2}{\sum bc}=\frac{\sum a^2+2\sum bc}{\sum bc}=2+\frac{\sum a^2}{\sum bc}\overset{(1)}{\ge}\frac{\lambda+6}{4}+\frac{\lambda abc}{\sum a^2b+abc}=RHS$$

where (1) $\Leftrightarrow 2+\frac{\sum a^2}{\sum bc}\ge\frac{\lambda+6}{4}+\frac{\lambda abc}{\sum a^2b+abc}\Leftrightarrow\frac{\sum a^2}{\sum bc}\ge\frac{\lambda-2}{4}+\frac{\lambda abc}{\sum a^2b+abc}$,

which follows from:

Lemma: For any $a,b,c>0, a+b+c=3$ then:

$$a^2b+b^2c+c^2a+abc\le 4$$

WLOG, we can suppose that $(b-a)(b-c)\le 0\Leftrightarrow b^2\le ab+bc-ac$

It suffices to prove that: $a^2b+(ab+bc-ac)c+c^2a+abc\le 4\Leftrightarrow b(a+c)^2\le 4$,

which follows from AM-GM:

$$\frac{b(a+c)^2}{4}=b\cdot\frac{a+c}{2}\cdot\frac{a+c}{2}\le\left(\frac{b+\frac{a+c}{2}+\frac{a+c}{2}}{3}\right)^3=\left(\frac{a+b+c}{3}\right)^3=\left(\frac{3}{3}\right)^3=1.$$

It remains to prove that:

$$\frac{\sum a^2}{\sum bc}\ge\frac{\lambda-2}{4}+\frac{\lambda abc}{4}$$

which follows from:

$$\frac{\sum a^2}{\sum bc}\ge 1$$

It suffices to prove that:

$$1\ge\frac{\lambda-2}{4}+\frac{\lambda abc}{4}\Leftrightarrow 1-\frac{\lambda-2}{4}\ge\frac{\lambda abc}{4}\Leftrightarrow\frac{6-\lambda}{4}\ge\frac{\lambda abc}{4}\Leftrightarrow\frac{6-\lambda}{\lambda}\ge abc$$

which follows from:

$$\frac{6-\lambda}{\lambda}\overset{0\le\lambda\le 3}{\ge}1=\left(\frac{a+b+c}{3}\right)^3\overset{AGM}{\ge}abc$$

Equality holds if and only if $a=b=c=1$.