**JP.574 In $\Delta ABC$ the following relationship holds:**

$$\frac{27}{8}\le\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}\ge\frac{27R}{16r}$$

*Proposed by Marin Chirciu – Romania*

*Solution by proposer*

**Lemma:** In $\Delta ABC$:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}=\frac{4p^2}{p^2+r^2+2Rr}$$

**Proof:**

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}=\frac{(2p)^2}{\sum a\sum bc-abc}=$$

$$=\frac{8p^3}{2p(p^2+r^2+4Rr)-4Rrp}=\frac{4p^2}{p^2+r^2+2Rr}$$

Let's get back to the main problem.Using the Lemma we obtain:

Right hand inequality:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}=\frac{4p^2}{p^2+r^2+2Rr}\overset{(1)}{\le}\frac{27R}{16r},$$

where (1) $\Leftrightarrow \frac{4p^2}{p^2+r^2+2Rr}\le\frac{27R}{16r}\Leftrightarrow p^2(27R-16r)+27Rr(2R+r)\ge 0.$

We distinguish the cases:

Case 1. If $(27R-16r)\ge 0$ the inequality is obvious.

Case 2. If $(27R-16r)<0$ the inequality can be rewritten:

$27Rr(2R+r)\ge p^2(16r-27R)$, which follows from Gerretsen's inequality:

$$p^2\le 4R^2+4Rr+3r^2.$$

It remains to prove that:

$$27Rr(2R+r)\ge(4R^2+4Rr+3r^2)(16r-27R)\Leftrightarrow$$

$$54R^3-47R^2r-74Rr^2-96r^3\ge 0$$

$$\Leftrightarrow (R-2r)(54R^2+61Rr+48r^2)\ge 0, \text{ see } R\ge 2r, \text{ (Euler)}.$$

Equality holds if and only if the triangle is equilateral.

**Left hand inequality:**

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}=\frac{4p^2}{p^2+r^2+2Rr}\overset{(2)}{\geq}\frac{27}{8},$$

**where (2)** $\Leftrightarrow \frac{2p^2}{p^2+r^2+2Rr}\geq\frac{27}{8}\Leftrightarrow 5p^2\geq 27r(2R+r)$**, which follows from Gerretsen's inequality:** $p^2\geq 16Rr-5r^2$**. It remains to prove that:**

$$5(16Rr-5r^2)\geq 27r(2R+r)\Leftrightarrow 26R\geq 54r\Leftrightarrow R\geq 2r,\ \text{(Euler)}.$$

**Equality holds if and only if the triangle is equilateral.**