ROMANIAN MATHEMATICAL MAGAZINE

JP.573 If $a,b,c>0$ and $\lambda\ge\frac{1}{4}$ then:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\lambda\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\frac{3}{2}(2\lambda+1)$$

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Solution by proposer

With the substitution $(x,y,z)=\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ we have $xyz=1$,

$$\frac{a}{a+b}=\frac{a}{b\left(\frac{a}{b}+1\right)}=\frac{x}{x+1}=\frac{x}{x+xyz}=\frac{1}{1+yz}$$

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{xy+yz+zx}{xyz}=\frac{xy+yz+zx}{1}=xy+yz+zx$$

The problem can be reformulated:

If $x,y,z>0, xyz=1$ and $\lambda\ge\frac{1}{4}$ then:

$$\frac{1}{1+yz}+\frac{1}{1+zx}+\frac{1}{1+xy}+\lambda(xy+yz+zx)\ge\frac{3}{2}(2\lambda+1).$$

Proof:

Denoting $(yz,zx,xy)=(m,n,p)$ we obtain

$$\frac{1}{1+m}+\frac{1}{1+n}+\frac{1}{1+p}+\lambda(m+n+p)\ge\frac{3}{2}(2\lambda+1)$$

which follows from:

$$\frac{1}{1+m}+\frac{1}{1+n}+\frac{1}{1+p}+\lambda(m+n+p)\overset{CS}{\ge}\frac{(1+1+1)^2}{m+n+p+3}+\lambda(m+n+p)=$$

$$=\frac{9}{m+n+p+3}+\lambda(m+n+p)\overset{(1)}{\ge}\frac{3}{2}(2\lambda+1),$$

where (1) $\Leftrightarrow\frac{9}{m+n+p+3}+\lambda(m+n+p)\ge\frac{3}{2}(2\lambda+1)\overset{m+n+p=t}{\Leftrightarrow}\frac{9}{t+3}+\lambda t\ge\frac{3}{2}(2\lambda+1)\Leftrightarrow$

$\Leftrightarrow 2\lambda t^2-3t+9-18\lambda\ge0\Leftrightarrow(t-3)(2\lambda t+6\lambda-3)\ge0$, true from $t\ge3$ and $\lambda\ge\frac{1}{4}$,

see $t=m+n+p=xy+yz+zx\ge3\sqrt[3]{(xyz)^2}=3\sqrt[3]{1^2}=3$.

Equality holds if and only if $a=b=c$.