ROMANIAN MATHEMATICAL MAGAZINE

JP.572. If $a,b,c,d>0$ then:

$$a\sqrt{\frac{a}{a^3+3bcd}}+b\sqrt{\frac{b}{b^3+3cda}}+c\sqrt{\frac{c}{c^3+3dab}}+d\sqrt{\frac{d}{d^3+3abc}}\ge 2$$

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Solution by proposer

$$\text{Let be } f(0,\infty)\to\mathbb{R};\ f(x)=\frac{3\ln x}{16}+\frac{1}{\sqrt{1+3x}},\quad f'(x)=\frac{3}{16x}-\frac{3}{2(1+3x)\sqrt{1+3x}}$$

$$f'(x)=0\Rightarrow\frac{1}{8x}=\frac{1}{\left(\sqrt{1+3x}\right)^3}\Rightarrow\sqrt{(1+3x)^3}-8x=0$$

$$\text{Let be } g:(0,\infty)\to\mathbb{R};\ g(x)=\sqrt{(1+3x)^3}-8x$$

$$g'(x)=\frac{9}{2}\sqrt{1+3x}-8,\ \ g''(x)=\frac{9}{2}\cdot\frac{3}{2\sqrt{1+3x}}>0$$

$g'(x)$ strictly increasing $\Rightarrow g'$ - injective

$g'(x)=g'(1)=0\Rightarrow x=1$ the unique solution $f'(x)=0\Rightarrow x=1$

$$\left.\begin{array}{l} If\ x<1\ then\ f'(x)<0\\ If\ x>1\ then\ f'(x)>0\end{array}\right\}\Rightarrow f(x)\ge f(1)=\frac{1}{2}$$

$$\frac{3\ln x}{16}+\frac{1}{\sqrt{1+3x}}\ge\frac{1}{2};\ (\forall)x>0$$

$$\text{For } x=\frac{bcd}{a^3}\Rightarrow\frac{3}{16}\ln\left(\frac{bcd}{a^3}\right)+\frac{1}{\sqrt{1+\frac{3bcd}{a^3}}}\ge\frac{1}{2}\quad (1)$$

$$\text{For } x=\frac{cda}{b^3}\Rightarrow\frac{3}{16}\ln\left(\frac{cda}{b^3}\right)+\frac{1}{\sqrt{1+\frac{3bcd}{b^3}}}\ge\frac{1}{2}\quad (2)$$

$$\text{For } x=\frac{dab}{c^3}\Rightarrow\frac{3}{16}\ln\left(\frac{dab}{c^3}\right)+\frac{1}{\sqrt{1+\frac{3dab}{c^3}}}\ge\frac{1}{2}\quad (3)$$

$$\text{For } x=\frac{abc}{d^3}\Rightarrow\frac{3}{16}\ln\left(\frac{abc}{d^3}\right)+\frac{1}{\sqrt{1+\frac{3abc}{d^3}}}\ge\frac{1}{2}\quad (4)$$

By adding (1); (2); (3); (4):

$$\frac{3}{16}\ln\left(\frac{bcd}{a^3}\cdot\frac{cda}{b^3}\cdot\frac{dab}{c^3}\cdot\frac{abc}{d^3}\right)+\sum_{cyc}\frac{1}{\sqrt{1+\frac{3bcd}{a^3}}}\ge\frac{1}{2}\cdot 4$$

$$\frac{3}{16}\ln 1+\sum_{cyc}\sqrt{\frac{a^3}{a^3+3bcd}}\ge 2, \qquad \sum_{cyc}a\sqrt{\frac{a}{a^3+3bcd}}\ge 2$$

Equality holds for: $a=b=c=d$.