ROMANIAN MATHEMATICAL MAGAZINE

JP.571 If $x, y, z > 0, x + y + z = 3$ then find the minimum value of

$$P = \sum \frac{y+z}{x^3+2} + \frac{1}{2}\sum x^2$$

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Solution by proposer

Lemma:

If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{y+z}{x^3+2} \ge \frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)$$

Proof:

We have

$$\frac{y+z}{x^3+2} = \frac{y+z}{2}\left(\frac{x^3+2-x^3}{x^3+2}\right) = \frac{y+z}{2}\left(1 - \frac{x^3}{x^3+2}\right) = \frac{y+z}{2}\left(1 - \frac{x^3}{x^3+1+1}\right) \overset{AGM}{\ge}$$

$$\overset{AGM}{\ge} \frac{y+z}{2}\left(1 - \frac{x^3}{3\sqrt[3]{x^3 \cdot 1 \cdot 1}}\right) =$$

$$= \frac{y+z}{2}\left(1 - \frac{x^3}{3x}\right) = \frac{y+z}{2}\left(1 - \frac{x^2}{3}\right) = \frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)$$

with equality for $x = 1$. Let's get back to the main problem.

Using the Lemma, we obtain:

$$P = \sum \frac{y+z}{x^3+2} + \frac{1}{2}\sum x^2 \overset{Lemma}{\ge} \sum\left(\frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)\right) + \frac{1}{2}\sum x^2 =$$

$$= \sum x - \frac{1}{6}\sum x^2(3-x) + \frac{1}{2}\sum x^2 =$$

$$= 3 + \frac{1}{6}\sum x^3 - \frac{1}{2}\sum x^2 + \frac{1}{2}\sum x^2 \overset{Holder}{\ge} 3 + \frac{1}{6} \cdot \frac{(\sum x)^3}{9} = 3 + \frac{1}{6} \cdot \frac{(3)^3}{9} = \frac{7}{2}$$

It follows that $P = \frac{7}{2}$ and the minimum is touched for $(x, y, z) = (1, 1, 1)$.