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A SIMPLE PROOF FOR BLUNDON-ROUCHE'S INEQUALITY

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Dedicated to Zaicanu Nicoleta Cristina

In ΔABC denote I –the incenter, O –the circumcenter, N_a –the Nagel's point.

It is known that:

$$OI = \sqrt{\mathrm{R}(\mathrm{R}-2\mathrm{r})}, ON_a = R - 2r, IN_a = \sqrt{\mathrm{s}^2 + 5\mathrm{r}^2 - 16\mathrm{Rr}}$$

In ΔOIN_a:

$$|OI - ON_a| \leq IN_a \leq OI + ON_a$$

$$|OI - ON_a| \leq IN_a \Leftrightarrow \left|\sqrt{R(R-2r)} - (R-2r)\right| \leq \sqrt{s^2 + 5r^2 - 16Rr}$$

By squaring:

$$R(R-2r) - 2(R-2r)\sqrt{R(R-2r)} + (R-2r)^2 \leq s^2 + 5r^2 - 16Rr$$

$$R^2 - 2rR - 2(R-2r)\sqrt{R(R-2r)} + R^2 - 4rR + 4r^2 \leq s^2 + 5r^2 - 16Rr$$

$$\boxed{\mathbf{2R^2 + 10Rr - 2(R-2r)\sqrt{R(R-2r)} - r^2 \leq s^2}}$$

$$IN_a \leq OI + ON_a \Leftrightarrow \sqrt{s^2 + 5r^2 - 16Rr} \leq \sqrt{R(R-2r)} + R - 2r$$

By squaring:

$$s^2 + 5r^2 - 16Rr \leq R(R-2r) + 2(R-2r)\sqrt{R(R-2r)} + (R-2r)^2$$

$$s^2 + 5r^2 - 16Rr \leq R^2 - 2rR + 2(R-2r)\sqrt{R(R-2r)} + R^2 - 4rR + 4r^2$$

$$\boxed{\mathbf{s^2 \leq 2R^2 + 10Rr + 2(R-2r)\sqrt{R(R-2r)} - r^2}}$$

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro