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SOLUTIONS

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PROBLEMS FOR JUNIORS

JP.571 If $x, y, z > 0, x + y + z = 3$ then find the minimum value of

$$P = \sum \frac{y+z}{x^3+2} + \frac{1}{2}\sum x^2$$

Proposed by Marin Chirciu – Romania

Solution by proposer

Lemma:

If $x, y, z > 0, x + y + z = 3$ then:

$$\frac{y+z}{x^3+2} \geq \frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)$$

Proof:

We have

$$\frac{y+z}{x^3+2} = \frac{y+z}{2}\left(\frac{x^3+2-x^3}{x^3+2}\right) = \frac{y+z}{2}\left(1 - \frac{x^3}{x^3+2}\right) = \frac{y+z}{2}\left(1 - \frac{x^3}{x^3+1+1}\right) \overset{AGM}{\geq}$$

$$\overset{AGM}{\geq} \frac{y+z}{2}\left(1 - \frac{x^3}{3\sqrt[3]{x^3 \cdot 1 \cdot 1}}\right) =$$

$$= \frac{y+z}{2}\left(1 - \frac{x^3}{3x}\right) = \frac{y+z}{2}\left(1 - \frac{x^2}{3}\right) = \frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)$$

with equality for $x = 1$. Let's get back to the main problem.

Using the Lemma, we obtain:

$$P = \sum \frac{y+z}{x^3+2} + \frac{1}{2}\sum x^2 \overset{Lemma}{\geq} \sum\left(\frac{1}{2}(y+z) - \frac{1}{6}x^2(y+z)\right) + \frac{1}{2}\sum x^2 =$$

$$= \sum x - \frac{1}{6}\sum x^2(3-x) + \frac{1}{2}\sum x^2 =$$

$$= 3 + \frac{1}{6}\sum x^3 - \frac{1}{2}\sum x^2 + \frac{1}{2}\sum x^2 \overset{Holder}{\geq} 3 + \frac{1}{6}\cdot\frac{(\sum x)^3}{9} = 3 + \frac{1}{6}\cdot\frac{(3)^3}{9} = \frac{7}{2}$$

It follows that $P = \frac{7}{2}$ and the minimum is touched for $(x, y, z) = (1, 1, 1)$.

JP.572 If $a,b,c,d>0$ then:

$$a\sqrt{\frac{a}{a^3+3bcd}}+b\sqrt{\frac{b}{b^3+3cda}}+c\sqrt{\frac{c}{c^3+3dab}}+d\sqrt{\frac{d}{d^3+3abc}}\ge 2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be $f(0,\infty)\to\mathbb{R};f(x)=\frac{3\ln x}{16}+\frac{1}{\sqrt{1+3x}},\quad f'(x)=\frac{3}{16x}-\frac{3}{2(1+3x)\sqrt{1+3x}}$

$$f'(x)=0\Rightarrow\frac{1}{8x}=\frac{1}{\left(\sqrt{1+3x}\right)^3}\Rightarrow\sqrt{(1+3x)^3}-8x=0$$

Let be $g:(0,\infty)\to\mathbb{R};g(x)=\sqrt{(1+3x)^3}-8x$

$$g'(x)=\frac{9}{2}\sqrt{1+3x}-8,\quad g''(x)=\frac{9}{2}\cdot\frac{3}{2\sqrt{1+3x}}>0$$

$g'(x)$ strictly increasing $\Rightarrow g'$ - injective

$g'(x)=g'(1)=0\Rightarrow x=1$ the unique solution $f'(x)=0\Rightarrow x=1$

$$\left.\begin{array}{l}If\ x<1\ then\ f'(x)<0\\ If\ x>1\ then\ f'(x)>0\end{array}\right\}\Rightarrow f(x)\ge f(1)=\frac{1}{2}$$

$$\frac{3\ln x}{16}+\frac{1}{\sqrt{1+3x}}\ge\frac{1}{2};(\forall)x>0$$

For $x=\frac{bcd}{a^3}\Rightarrow\frac{3}{16}\ln\left(\frac{bcd}{a^3}\right)+\frac{1}{\sqrt{1+\frac{3bcd}{a^3}}}\ge\frac{1}{2}$ (1)

For $x=\frac{cda}{b^3}\Rightarrow\frac{3}{16}\ln\left(\frac{cda}{b^3}\right)+\frac{1}{\sqrt{1+\frac{3bcd}{b^3}}}\ge\frac{1}{2}$ (2)

For $x=\frac{dab}{c^3}\Rightarrow\frac{3}{16}\ln\left(\frac{dab}{c^3}\right)+\frac{1}{\sqrt{1+\frac{3dab}{c^3}}}\ge\frac{1}{2}$ (3)

For $x=\frac{abc}{d^3}\Rightarrow\frac{3}{16}\ln\left(\frac{abc}{d^3}\right)+\frac{1}{\sqrt{1+\frac{3abc}{d^3}}}\ge\frac{1}{2}$ (4)

By adding (1); (2); (3); (4):

$$\frac{3}{16}\ln\left(\frac{bcd}{a^3}\cdot\frac{cda}{b^3}\cdot\frac{dab}{c^3}\cdot\frac{abc}{d^3}\right)+\sum_{cyc}\frac{1}{\sqrt{1+\frac{3bcd}{a^3}}}\ge\frac{1}{2}\cdot 4$$

$$\frac{3}{16}\ln 1+\sum_{cyc}\sqrt{\frac{a^3}{a^3+3bcd}}\geq 2,\qquad \sum_{cyc}a\sqrt{\frac{a}{a^3+3bcd}}\geq 2$$

Equality holds for: $a=b=c=d$.

JP.573 If $a,b,c>0$ and $\lambda\geq\frac{1}{4}$ then:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\lambda\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq\frac{3}{2}(2\lambda+1)$$

Proposed by Marin Chirciu – Romania

Solution by proposer

With the substitution $(x,y,z)=\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ we have $xyz=1$,

$$\frac{a}{a+b}=\frac{a}{b\left(\frac{a}{b}+1\right)}=\frac{x}{x+1}=\frac{x}{x+xyz}=\frac{1}{1+yz}$$

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{xy+yz+zx}{xyz}=\frac{xy+yz+zx}{1}=xy+yz+zx$$

The problem can be reformulated:

If $x,y,z>0, xyz=1$ and $\lambda\geq\frac{1}{4}$ then:

$$\frac{1}{1+yz}+\frac{1}{1+zx}+\frac{1}{1+xy}+\lambda(xy+yz+zx)\geq\frac{3}{2}(2\lambda+1).$$

Proof:

Denoting $(yz,zx,xy)=(m,n,p)$ we obtain

$$\frac{1}{1+m}+\frac{1}{1+n}+\frac{1}{1+p}+\lambda(m+n+p)\geq\frac{3}{2}(2\lambda+1)$$

which follows from:

$$\frac{1}{1+m}+\frac{1}{1+n}+\frac{1}{1+p}+\lambda(m+n+p)\overset{CS}{\geq}\frac{(1+1+1)^2}{m+n+p+3}+\lambda(m+n+p)=$$

$$=\frac{9}{m+n+p+3}+\lambda(m+n+p)\overset{(1)}{\geq}\frac{3}{2}(2\lambda+1),$$

where (1) $\Leftrightarrow\frac{9}{m+n+p+3}+\lambda(m+n+p)\geq\frac{3}{2}(2\lambda+1)\overset{m+n+p=t}{\Leftrightarrow}\frac{9}{t+3}+\lambda t\geq\frac{3}{2}(2\lambda+1)\Leftrightarrow$

$\Leftrightarrow 2\lambda t^2-3t+9-18\lambda\geq 0\Leftrightarrow(t-3)(2\lambda t+6\lambda-3)\geq 0$, true from $t\geq 3$ and $\lambda\geq\frac{1}{4}$, see $t = m + n + p = xy + yz + zx \geq 3\sqrt[3]{(xyz)^2} = 3\sqrt[3]{1^2} = 3$.

Equality holds if and only if $a = b = c$.

JP.574 In ΔABC the following relationship holds:

$$\frac{27}{8} \leq \frac{(a+b+c)^3}{(a+b)(b+c)(c+a)} \geq \frac{27R}{16r}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

Lemma: In ΔABC:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)} = \frac{4p^2}{p^2 + r^2 + 2Rr}$$

Proof:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)} = \frac{(2p)^2}{\sum a \sum bc - abc} =$$

$$= \frac{8p^3}{2p(p^2 + r^2 + 4Rr) - 4Rrp} = \frac{4p^2}{p^2 + r^2 + 2Rr}$$

Let's get back to the main problem.Using the Lemma we obtain:

Right hand inequality:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)} = \frac{4p^2}{p^2 + r^2 + 2Rr} \overset{(1)}{\leq} \frac{27R}{16r},$$

where (1) $\Leftrightarrow \frac{4p^2}{p^2+r^2+2Rr} \leq \frac{27R}{16r} \Leftrightarrow p^2(27R - 16r) + 27Rr(2R + r) \geq 0$.

We distinguish the cases:

Case 1. If $(27R - 16r) \geq 0$ the inequality is obvious.

Case 2. If $(27R - 16r) < 0$ the inequality can be rewritten:

$27Rr(2R + r) \geq p^2(16r - 27R)$, which follows from Gerretsen's inequality:

$$p^2 \leq 4R^2 + 4Rr + 3r^2.$$

It remains to prove that:

$$27Rr(2R + r) \geq (4R^2 + 4Rr + 3r^2)(16r - 27R) \Leftrightarrow$$

$$54R^3 - 47R^2 r - 74Rr^2 - 96r^3 \geq 0$$

$$\Leftrightarrow (R-2r)(54R^2+61Rr+48r^2)\geq 0, \text{ see } R\geq 2r, \text{ (Euler).}$$

Equality holds if and only if the triangle is equilateral.

Left hand inequality:

$$\frac{(a+b+c)^3}{(a+b)(b+c)(c+a)}=\frac{4p^2}{p^2+r^2+2Rr}\overset{(2)}{\geq}\frac{27}{8},$$

where (2) $\Leftrightarrow \frac{2p^2}{p^2+r^2+2Rr}\geq\frac{27}{8}\Leftrightarrow 5p^2\geq 27r(2R+r)$, which follows from Gerretsen's inequality: $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$5(16Rr-5r^2)\geq 27r(2R+r)\Leftrightarrow 26R\geq 54r\Leftrightarrow R\geq 2r, \text{ (Euler).}$$

Equality holds if and only if the triangle is equilateral.

JP.575 Find $x,y\in\mathbb{R}$ such that:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=10$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marin Chirciu-Romania

Using Minkowski inequality we obtain:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}=\sqrt{x^2+y^2}+\sqrt{(4-x)^2+(3-y)^2}\geq$$
$$\geq\sqrt{(x+4-x)^2+(y+3-y)^2}=5$$

with equal for $\frac{x}{y}=\frac{4-x}{3-y}$.

$$\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=\sqrt{(4-x)^2+y^2}+\sqrt{x^2+(3-y)^2}\geq$$
$$\geq\sqrt{(4-x+x)^2+(y+3-y)^2}=5$$

with equal for $\frac{4-x}{y}=\frac{x}{3-y}$.

Adding the two equalities we obtain:

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}\geq 10$$

with equal for: $\frac{x}{y}=\frac{4-x}{3-y}$ and $\frac{4-x}{y}=\frac{x}{3-y}\Leftrightarrow(x,y)=\left(2,\frac{3}{2}\right)$.

We deduce that the solution of the equation is $(x,y)=\left(2,\frac{3}{2}\right)$.

Remark: The problem can be developed. Let be $\lambda\geq 0$ fixed. Being $x,y\in\mathbb{R}$ such that:

$$\sqrt{x^2+y^2}+\sqrt{(x-4\lambda)^2+(y-3\lambda)^2}+\sqrt{(x-4\lambda)^2+y^2}+\sqrt{x^2+(y-3\lambda)^2}=10\lambda$$

Marin Chirciu

Solution: **Using Minkowski inequality we obtain:**

$$\sqrt{x^2+y^2}+\sqrt{(x-4\lambda)^2+(y-3\lambda)^2}=\sqrt{x^2+y^2}+\sqrt{(4\lambda-x)^2+(3\lambda-y)^2}\geq$$

$$\geq\sqrt{(x+4\lambda-x)^2+(y+3\lambda-y)^2}=5\lambda, \text{ with equal for } \frac{x}{y}=\frac{4\lambda-x}{3\lambda-y}.$$

$$\sqrt{(x-4\lambda)^2+y^2}+\sqrt{x^2+(y-3\lambda)^2}=\sqrt{(4\lambda-x)^2+y^2}+\sqrt{x^2+(3\lambda-y)^2}\geq$$

$$\geq\sqrt{(4\lambda-x+x)^2+(y+3\lambda-y)^2}=5\lambda,$$

with equal for $\frac{4\lambda-x}{y}=\frac{x}{3\lambda-y}$**.Adding the two equality we obtain:**

$$\sqrt{x^2+y^2}+\sqrt{(x-4\lambda)^2+(y-3\lambda)^2}+\sqrt{(x-4\lambda)^2+y^2}+\sqrt{x^2+(y-3\lambda)^2}\geq 10\lambda$$

with equal for $\frac{x}{y}=\frac{4\lambda-x}{3\lambda-y}$ **and** $\frac{4\lambda-x}{y}=\frac{x}{3\lambda-y}\Leftrightarrow(x,y)=\left(2\lambda,\frac{3\lambda}{2}\right)$.

We deduce that the solution of the equation is $(x,y)=\left(2\lambda,\frac{3\lambda}{2}\right)$.

Note: For $\lambda=1$ **we obtain Problem JP.575 from RMM Number 39, Winter 2025.**

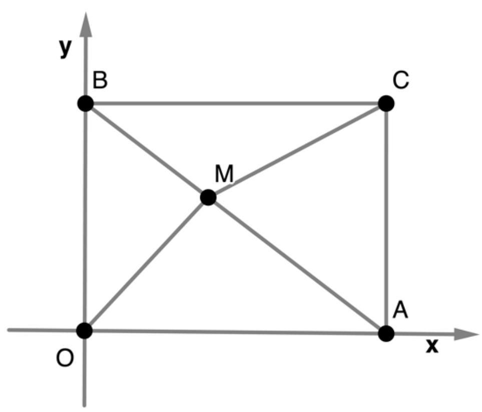
JP.575. Find $x,y\in\mathbb{R}$ **such that:**

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=10.$$

Daniel Sitaru – Romania

Solution 2 by proposer

Let be $A(4,0);B(0,3);C(4,3);O(0,0)$



$$MO=\sqrt{x^2+y^2};MC=\sqrt{(x-4)^2+(y-3)^2}$$

$$AB=\sqrt{4^2+3^2}=5;MA=\sqrt{(x-4)^2+y^2}$$

$$MB=\sqrt{x^2+(y-3)^2};CO=\sqrt{4^2+3^2}=5$$

In ΔMOC: $MO+MC\geq OC$

In ΔMAB: $MA+MB\geq AB$

By adding:

$$MO+MC+MA+MB\geq OC+AB$$

By adding:

$$MO+MC+MA+MB\geq OC+AB$$

$$\sqrt{x^2+y^2}+\sqrt{(x-4)^2+(y-3)^2}+\sqrt{(x-4)^2+y^2}+\sqrt{x^2+(y-3)^2}=10$$

$$MO+MC+MA+MB=OC+AB$$

Equality holds for $\{M\}=OC\cap AB$

$$M\left(\frac{x_A+x_B}{2};\frac{y_A+y_B}{2}\right)=\left(\frac{4+0}{2};\frac{0+3}{2}\right)=\left(2,\frac{3}{2}\right)\Rightarrow x=2;y=\frac{3}{2}$$

JP.576 If $a,b,c>0, a+b+c=3$ and $0\leq\lambda\leq 3$ then:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+\frac{\lambda abc}{a^2b+b^2c+c^2a+abc}\geq\frac{\lambda+6}{4}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

$$LHS=\sum\frac{a^2}{bc}\overset{CS}{\geq}\frac{(\sum a)^2}{\sum bc}=\frac{\sum a^2+2\sum bc}{\sum bc}=2+\frac{\sum a^2}{\sum bc}\overset{(1)}{\geq}\frac{\lambda+6}{4}+\frac{\lambda abc}{\sum a^2b+abc}=RHS$$

where (1) $\Leftrightarrow 2+\frac{\sum a^2}{\sum bc}\geq\frac{\lambda+6}{4}+\frac{\lambda abc}{\sum a^2b+abc}\Leftrightarrow\frac{\sum a^2}{\sum bc}\geq\frac{\lambda-2}{4}+\frac{\lambda abc}{\sum a^2b+abc}$,

which follows from:

Lemma: For any $a,b,c>0, a+b+c=3$ then:

$$a^2b+b^2c+c^2a+abc\leq 4$$

WLOG, we can suppose that $(b-a)(b-c)\leq 0\Leftrightarrow b^2\leq ab+bc-ac$

It suffices to prove that: $a^2b+(ab+bc-ac)c+c^2a+abc\le 4\Leftrightarrow b(a+c)^2\le 4$,

which follows from AM-GM:

$$\frac{b(a+c)^2}{4}=b\cdot\frac{a+c}{2}\cdot\frac{a+c}{2}\le\left(\frac{b+\frac{a+c}{2}+\frac{a+c}{2}}{3}\right)^3=\left(\frac{a+b+c}{3}\right)^3=\left(\frac{3}{3}\right)^3=1.$$

It remains to prove that:

$$\frac{\sum a^2}{\sum bc}\ge\frac{\lambda-2}{4}+\frac{\lambda abc}{4}$$

which follows from:

$$\frac{\sum a^2}{\sum bc}\ge 1$$

It suffices to prove that:

$$1\ge\frac{\lambda-2}{4}+\frac{\lambda abc}{4}\Leftrightarrow 1-\frac{\lambda-2}{4}\ge\frac{\lambda abc}{4}\Leftrightarrow\frac{6-\lambda}{4}\ge\frac{\lambda abc}{4}\Leftrightarrow\frac{6-\lambda}{\lambda}\ge abc$$

which follows from:

$$\frac{6-\lambda}{\lambda}\overset{0\le\lambda\le 3}{\ge}1=\left(\frac{a+b+c}{3}\right)^3\overset{AGM}{\ge}abc$$

Equality holds if and only if $a=b=c=1$.

JP.577 If $x,y,z>0, xyz=1$ **and** $n\in\mathbb{N}^*$ **then:**

$$\sum\frac{x^{2n+1}}{x+y+1}\ge 1$$

Proposed by Marin Chirciu – Romania

Solution by proposer

Lemma: If $x,y,z>0, xyz=1$ **and** $k\in\mathbb{N}$ **then:**

$$\sum x^{k+1}\ge\sum x^k$$

Proof:

Using Chebyshev's inequality we obtain:

$$\sum x^{k+1}=\sum x\cdot x^k\overset{Chebyshev}{\ge}\frac{1}{3}\sum x\sum x^k\overset{AGM}{\ge}\sqrt[3]{xyz}\cdot\sum x^k=\sqrt[3]{1}\cdot\sum x^k=\sum x^k$$

with equality for $x=y=z=1$. Let's get back to the main problem.

Using Lemma we obtain:

$$LHS=\sum\frac{x^{2n+1}}{x+y+1}=\sum\frac{x^{2n+2}}{x^2+xy+x}\overset{CS}{\geq}\frac{(\sum x^{n+1})^2}{\sum(x^2+xy+x)}=\frac{(\sum x^{n+1})^2}{\sum x^2+\sum xy+\sum x}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{(\sum x^{n+1})^2}{\sum x^{n+1}+\sum x^{n+1}+\sum x^{n+1}}=\frac{(\sum x^{n+1})^2}{3\sum x^{n+1}}=\frac{\sum x^{n+1}}{3}\overset{(1)}{\geq}1=RHD$$

where (1) $\Leftrightarrow \sum x^{n+1}\geq\sum x^2, \sum x^{n+1}\geq\sum x^2\geq\sum xy, \sum x^{n+1}\geq\sum x, \sum x^{n+1}\geq 3$, see Lemma. Equality holds if and only if $x=y=z=1$.

Note.

For $n=2$ we obtain the Proposed problem by Kostantinos Geronikolas in Mathematical Inequalities 2/2022: If $x,y,z>0, xyz=1$ then:

$$\sum\frac{x^5}{x+y+1}\geq 1$$

Konstantinos Geronikolas - Greece

JP.578 If $x,y,z>0$ and $n\in\mathbb{N}, n\geq 2$, in ΔABC holds:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}\geq\frac{\sqrt{3}}{2r}(6r^2)^n$$

Proposed by Marin Chirciu – Romania

Solution by proposer

Lemma:

If $x,y,z>0$ and $n\in\mathbb{N}, n\geq 2$, in ΔABC holds:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}\geq\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n$$

Proof:

$$\sum\frac{x^n a^{2n-1}}{(y+z)^n}=\sum\frac{\left(\frac{x}{y+z}a^2\right)^n}{a}\overset{Holder}{\geq}\frac{\left(\sum\frac{x}{y+z}a^2\right)^n}{3^{n-2}\sum a}\overset{Tsintsifas}{\geq}\frac{(2\sqrt{3}F)^n}{3^{n-2}\cdot 2p}=$$

$$=\frac{9}{2p}\left(\frac{2\sqrt{3}F}{3}\right)^n=\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n$$

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS=\sum\frac{x^{2n}a^{2n+1}}{(y+z)^{2n}}\overset{Lemma}{\geq}\frac{9}{2p}\left(\frac{2F}{\sqrt{3}}\right)^n=\frac{9}{2p}\cdot\frac{2^np^nr^n}{\left(\sqrt{3}\right)^n}=\frac{9}{2}\cdot\frac{2^np^{n-1}r^n}{\left(\sqrt{3}\right)^n}\geq$$

$$\overset{Mitrinovic}{\geq}\frac{9}{2}\cdot\frac{2^n\left(3\sqrt{3}r\right)^{n-1}r^n}{\left(\sqrt{3}\right)^n}=$$

$$=\frac{3^2}{2}\cdot\frac{2^n(3r)^{n-1}r^n}{\sqrt{3}}=\frac{3\sqrt{3}}{2}\cdot\frac{2^n\cdot 3^nr^{n-1}r^n}{3}=\frac{\sqrt{3}}{2}\cdot 6^n\cdot r^{2n-1}=\frac{\sqrt{3}}{2r}(6r^2)^n=RHS$$

Equality holds if and only if the triangle is equilateral. We have used above:

Lemma Tsintsifas:

If $x,y,z>0$ then in ΔABC holds:

$$\sum\frac{x}{y+z}a^2\geq 2\sqrt{3}F$$

Proof:

$$\sum\frac{x}{y+z}a^2=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\frac{x+y+z}{y+z}a^2-\sum a^2\overset{Bergstrom}{\geq}$$

$$\geq(x+y+z)\frac{(\sum a)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{(2p)^2}{2(x+y+z)}-2(p^2-r^2-4Rr)=$$

$$=2p^2-2(p^2-r^2-4Rr)=2(r^2+4Rr)=2r(4R+r).$$

We have used above the known identities in the triangle:

$$\sum a=2p \text{ and } \sum a^2=2(p^2-r^2-4Rr)$$

It remains to prove that: $2r(4R+r)\geq 2\sqrt{3}S\Leftrightarrow r(4R+r)\geq\sqrt{3}rp\Leftrightarrow 4R+r\geq p\sqrt{3}$,

which is Doucet's inequality.Equality holds if and only if the triangle is equilateral.

JP.579 Solve for real numbers:

$$\begin{cases}\dfrac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x-y)+yz(y-z)+zx(z-x)}=55\\ x^3+y^3+z^3=99\\ \dfrac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}=\dfrac{55}{9}\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Denote: $S_1=x+y+z; S_2=xy+yz+zx; S_3=xyz.$

$$\frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x-y)+yz(y-z)+zx(z-x)}=55$$

$$\frac{(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)}{(x-y)(y-z)(z-x)}=55$$

$$x^2+y^2+z^2+xy+yz+zx=55,\qquad S_1^2-2S_2+S_2=55\Rightarrow S_1^2+S_2=55$$

$$\frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)}=\frac{55}{9}$$

$$\frac{x^2+y^2+z^2+xy+yz+zx}{x+y+z}=\frac{55}{9}$$

$$\frac{S_1^2-S_2}{S_1}=\frac{55}{9}\Rightarrow\frac{55}{S_1}=\frac{55}{9}\Rightarrow S_1=9$$

$$S_1^2-S_2=55\Rightarrow 9^2-S_2=55\Rightarrow 81-S_2=55\Rightarrow S_2=26$$

$$x^3+y^3+z^3=99,\qquad S_1\left(S_1^2-3S_2\right)+3S_3=99$$

$$9(81-3\cdot 26)+3S_3=99\Rightarrow 9(81-78)+3S_3=99$$

$$3S_3=99-27\Rightarrow S_3=33-9\Rightarrow S_3=24$$

Let be the equation with roots x,y,z:

$$u^3-S_1u^2+S_2u-S_3=0$$

$$u^3-9u^2+26u-25=0$$

$$(u-2)(u-3)(u-4)=0\Rightarrow u_1=2;u_2=3;u_3=4$$

Solutions: $(x,y,z)=(2,3,4)$ and permutations.

JP.580 If $a,b\geq 0; x\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{[2^a+\sin^2x]+[2^b+\cos^2x]}{[2^a+2^b]}+\frac{[2^b+\sin^2x]+[2^c+\cos^2x]}{[2^b+2^c]}+\frac{[2^c+\sin^2x]+[2^a+\cos^2x]}{[2^c+2^a]}\geq 3$$

[*] – great integer function.

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$2^a+\sin^2 x-1<[2^a+\sin^2 x]$$
$$2^b+\cos^2 x-1<[2^b+\cos^2 x]$$

By adding:

$$2^a+2^b+\sin^2 x+\cos^2 x-2<[2^a+\sin^2 x]+[2^b+\cos^2 x]$$
$$2^a+2^b+1-2<[2^a+\sin^2 x]+[2^b+\cos^2 x]$$
$$2^a+2^b-1<[2^a+\sin^2 x]+[2^b+\cos^2 x]$$
$$[2^a+2^b-1]<\left[[2^a+\sin^2 x]+[2^b+\cos^2 x]\right]$$
$$[2^a+2^b]\le[2^a+\sin^2 x]+[2^b+\cos^2 x]$$
$$\frac{[2^a+\sin^2 x]+[2^b+\cos^2 x]}{[2^a+2^b]}\ge 1 \quad (1)$$

Analogous:

$$\frac{[2^b+\sin^2 x]+[2^c+\cos^2 x]}{[2^b+2^c]}\ge 1 \quad (2)$$
$$\frac{[2^c+\sin^2 x]+[2^a+\cos^2 x]}{[2^c+2^a]}\ge 1 \quad (3)$$

By adding (1); (2); (3):

$$\frac{[2^a+\sin^2 x]+[2^b+\cos^2 x]}{[2^a+2^b]}+\frac{[2^b+\sin^2 x]+[2^c+\cos^2 x]}{[2^b+2^c]}+$$
$$+\frac{[2^c+\sin^2 x]+[2^a+\cos^2 x]}{[2^c+2^a]}\ge 3$$

JP.581 If $x,y,z\ge 0$ then:

$$\frac{[3^x]\cdot[3^y]}{[3^{x+y}]}+\frac{[3^y]\cdot[3^z]}{[3^{y+z}]}+\frac{[3^z]\cdot[3^x]}{[3^{z+x}]}\le 3$$

$[*]$ - great integer function.

Proposed by Daniel Sitaru – Romania

Solution by proposer

Lemma: If $a,b\in[0,\infty)$ then:

$$[a]\cdot[b]\le[a\cdot b] \quad (1)$$

Proof:

$$[a] \le a; [b] \le b \Rightarrow [a] \cdot [b] \le ab \Rightarrow \big[[a] \cdot [b]\big] \le [a \cdot b] \Rightarrow [a] \cdot [b] \le [a \cdot b]$$

Back to the problem:

We take in (1): $a = 3^x; b = 3^y$

$$[3^x] \cdot [3^y] \le [3^x \cdot 3^y] \Rightarrow \frac{[3^x] \cdot [3^y]}{[3^{x+y}]} \le 1 \quad (2)$$

Analogous:

$$\frac{[3^y] \cdot [3^z]}{[3^{y+z}]} \le 1 \quad (3)$$

$$\frac{[3^z] \cdot [3^x]}{[3^{z+x}]} \le 1 \quad (4)$$

By adding (2); (3); (4):

$$\frac{[3^x] \cdot [3^y]}{[3^{x+y}]} + \frac{[3^y] \cdot [3^z]}{[3^{y+z}]} + \frac{[3^z] \cdot [3^x]}{[3^{z+x}]} \le 3$$

Equality holds for $x, y, z \in \mathbb{N}$.

JP.582 **In acute $\Delta ABC; AA', BB', CC'$ - altitudes; H – orthocenter; m_a, m_b, m_c – medians; r – inradii. Prove that:**

$$\frac{m_a^2}{HA'} + \frac{m_b^2}{HB'} + \frac{m_c^2}{HC'} \ge \frac{81r^2(\tan A + \tan B + \tan C)}{h_a \tan A\, h_b \tan B + h_c \tan C}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Firstly}, \sum_{\text{cyc}} m_a \ge \sum_{\text{cyc}} h_a = \frac{s^2 + 4Rr + r^2}{2R} - 9r + 9r = \frac{s^2 - 14Rr + r^2}{2R} + 9r$$
$$= \frac{s^2 - 16Rr + 5r^2 + 2r(R - 2r)}{2R} + 9r \ge 9r$$
$$\left(\because s^2 - 16Rr + 5r^2 \overset{\text{Gerretsen}}{\ge} 0 \text{ and } 2r(R - 2r) \overset{\text{Euler}}{\ge} 0\right) \therefore \sum_{\text{cyc}} m_a \ge 9r \rightarrow (1)$$

$$\text{Now}, HA' = AA' - AH = h_a - 2R\cos A = \frac{4R^2 \sin B \sin C}{2R} - 2R\cos A$$
$$= R(\cos(B - C) - \cos(B + C)) - 2R\cos A = R(\cos(B - C) + \cos A) - 2R\cos A$$
$$= R(\cos(B - C) - \cos A) = R(\cos(B - C) + \cos(B + C)) = 2R\cos B\cos C \therefore$$

$$\mathrm{HA}' = 2\mathrm{R}\cos \mathrm{B}\cos \mathrm{C} \ \textit{and analogs} \Rightarrow \mathrm{HA}' + \mathrm{HB}' + \mathrm{HC}' = 2\mathrm{R}\sum_{\mathrm{cyc}} \cos \mathrm{B}\cos \mathrm{C} \to (2)$$

$$\textit{Again}, \frac{81\mathrm{r}^2(\tan A + \tan B + \tan C)}{h_a \tan A + h_b \tan B + h_c \tan C} = \frac{81\mathrm{r}^2.\tan A \tan B \tan C}{\sum_{\mathrm{cyc}}\left(\frac{2\mathrm{rs}}{2\mathrm{R}\sin A}\cdot\frac{\sin A}{\cos A}\right)} = \frac{81\mathrm{r}^2.\frac{\prod_{\mathrm{cyc}}\sin A}{\prod_{\mathrm{cyc}}\cos A}}{\frac{\mathrm{rs}}{\mathrm{R}}\cdot\frac{\sum_{\mathrm{cyc}}\cos B\cos C}{\prod_{\mathrm{cyc}}\cos A}}$$

$$= \frac{81\mathrm{r}^2.\frac{4\mathrm{Rrs}}{8\mathrm{R}^3}}{\frac{\mathrm{rs}}{\mathrm{R}}\cdot\sum_{\mathrm{cyc}}\cos B\cos C} = \frac{81\mathrm{r}^2}{2\mathrm{R}\sum_{\mathrm{cyc}}\cos B\cos C}$$

$$\therefore \frac{81\mathrm{r}^2(\tan A + \tan B + \tan C)}{h_a \tan A + h_b \tan B + h_c \tan C} = \frac{81\mathrm{r}^2}{2\mathrm{R}\sum_{\mathrm{cyc}}\cos B\cos C} \to (3) \ \textit{and finally},$$

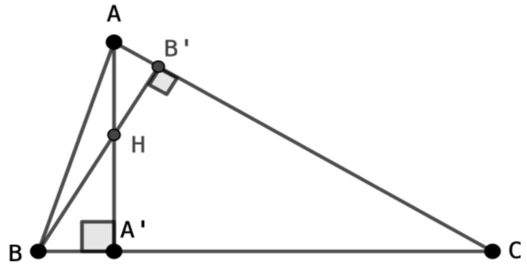
$$\frac{m_a^2}{\mathrm{HA}'} + \frac{m_b^2}{\mathrm{HB}'} + \frac{m_c^2}{\mathrm{HC}'} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} m_a\right)^2}{\mathrm{HA}' + \mathrm{HB}' + \mathrm{HC}'} \overset{\textit{via}\ (1)\ \text{and}\ (2)}{\geq}$$

$$\frac{81\mathrm{r}^2}{2\mathrm{R}\sum_{\mathrm{cyc}}\cos B\cos C} \overset{\textit{via}\ (3)}{=} \frac{81\mathrm{r}^2(\tan A + \tan B + \tan C)}{h_a \tan A + h_b \tan B + h_c \tan C}$$

$$\therefore \frac{m_a^2}{\mathrm{HA}'} + \frac{m_b^2}{\mathrm{HB}'} + \frac{m_c^2}{\mathrm{HC}'} \geq \frac{81\mathrm{r}^2(\tan A + \tan B + \tan C)}{h_a \tan A + h_b \tan B + h_c \tan C} \forall \Delta\,\mathrm{ABC},$$

$$'' = '' \text{ iff } \Delta\,\mathrm{ABC} \text{ is } \textit{equilateral} \ (\mathrm{QED})$$

Solution 2 by proposer



$$A'B = AB\cos B = c\cos B$$

$$m(\angle BHA') = m(\angle AHB') = 90^\circ - m(\angle A'AC) = 90^\circ - \left(90^\circ - m(\widehat{C})\right) = m(\widehat{C})$$

$$\tan(\angle BHA') = \frac{A'B}{HA'} \Rightarrow \tan C = \frac{A'B}{HA'}, \qquad HA' = \frac{A'B}{\tan C} = \frac{c\cos B}{\tan C} = \frac{2R\sin C\cdot\cos B}{\frac{\sin C}{\cos C}}$$

$$HA' = 2R\cos B\cos C, \qquad HA' = 2R\cdot\frac{1}{\frac{1}{\cos B\cos C}} = \frac{2R\sin B\sin C}{\frac{\sin B\sin C}{\cos B\cos C}}$$

$$HA' = \frac{2R\sin B\sin C}{\tan B\tan C} = \frac{2Ra\sin B\sin C}{a\tan B\tan C}, \qquad HA' = \frac{2R\cdot 2R\sin A\cdot\sin B\sin C}{a\tan B\tan C}$$

$$HA'=\frac{2\cdot 2R^2\sin A\sin B\sin C}{a\tan B\tan C}=\frac{2F}{a\tan B\tan C}$$

$$HA'=\frac{2\cdot\frac{a\cdot h_a}{2}}{a\tan B\tan C}=\frac{h_a}{\tan B\tan C}$$

$$HB'=\frac{h_b}{\tan C\tan A};HC'=\frac{h_c}{\tan A\tan B}$$

$$\frac{m_a^2}{HA'}+\frac{m_b^2}{HB'}+\frac{m_c^2}{HC'}\overset{BERGSTROM}{\geq}\frac{(m_a+m_b+m_c)^2}{HA'+HB'+HC'}=$$

$$=\frac{(m_a+m_b+m_c)^2}{\frac{h_a}{\tan B\tan C}+\frac{h_b}{\tan C\tan A}+\frac{h_c}{\tan A\tan B}}=$$

$$=\frac{(m_a+m_b+m_c)^2\cdot\tan A\tan B\tan C}{h_a\tan A+h_b\tan B+h_c\tan C}\overset{GOTMAN}{\leq}\frac{(9r)^2\cdot\tan A\tan B\tan C}{h_a\tan A+h_b\tan B+h_c\tan C}=$$

$$=\frac{81r^2(\tan A+\tan B+\tan C)}{h_a\tan A+h_b\tan B+h_c\tan C}$$

Equality holds for $a=b=c$.

JP.583 In ΔABC, a,b,c – sides, h_a,h_b,h_c – altitudes, s – semiperimeter, the following relationship holds:

$$\frac{h_a^2}{b^2+c^2}+\frac{h_b^2}{c^2+a^2}+\frac{h_c^2}{a^2+b^2}\leq\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{C-S}{\geq}9\ or,\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq\frac{9}{a+b+c}\ (1)$$

$$\sum\frac{h_a^2}{b^2+c^2}\leq\sum\frac{m_a^2}{b^2+c^2}=\frac{1}{4}\sum\frac{2b^2+2c^2-a^2}{b^2+c^2}=\frac{1}{4}\sum\left(2-\frac{a^2}{b^2+c^2}\right)=$$

$$=\frac{6}{4}-\frac{1}{4}\sum\left(\frac{a^2}{b^2+c^2}\right)\overset{Nesbitt}{\leq}\frac{6}{4}-\frac{1}{4}\cdot\frac{3}{2}=\frac{9}{8}=\frac{s}{4}\cdot\frac{9}{2s}=$$

$$=\frac{s}{4}\cdot\frac{9}{a+b+c}\overset{(1)}{\leq}\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Equality holds for an equilateral triangle.

Solution 2 by Marin Chirciu-Romania

$$LHS=\sum\frac{h_a^2}{b^2+c^2}\overset{AM-GM}{\le}\sum\frac{h_a^2}{2bc}=\frac{1}{2}\sum\frac{h_a^2}{bc}=\frac{1}{2}\cdot\frac{sr}{R}\sum\frac{1}{a}=\frac{sr}{2R}\sum\frac{1}{a}\overset{Euler}{\le}$$
$$\overset{Euler}{\le}\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=RHS.$$

We have used above:

$$\sum\frac{h_a^2}{bc}=\frac{4pr}{R}\sum\frac{1}{a}, \text{ see } \sum\frac{h_a^2}{bc}=\sum\frac{\frac{4F^2}{a^2}}{bc}=\frac{4F^2}{abc}\sum\frac{1}{a}=\frac{4s^2r^2}{4Rrs}\sum\frac{1}{a}=\frac{sr}{R}\sum\frac{1}{a}$$

Equality holds if and only if the triangle is equilateral.Remark: The problem cand be strengthened and developed.

In ΔABC:

$$\frac{9r^2}{2R^2}\le\sum\frac{h_a^2}{b^2+c^2}\le\frac{F}{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Marin Chirciu

Solution: Right hand inequality

$$LHS=\sum\frac{h_a^2}{b^2+c^2}\overset{AM-GM}{\le}\sum\frac{h_a^2}{2bc}=\frac{1}{2}\sum\frac{h_a^2}{bc}=\frac{1}{2}\cdot\frac{sr}{R}\sum\frac{1}{a}=\frac{sr}{2R}\sum\frac{1}{a}\overset{Euler}{\le}$$
$$\overset{Euler}{\le}\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=RHS.$$

We have used above:

$$\sum\frac{h_a^2}{bc}=\frac{4pr}{R}\sum\frac{1}{a}, \text{ see } \sum\frac{h_a^2}{bc}=\sum\frac{\frac{4F^2}{a^2}}{bc}=\frac{4F^2}{abc}\sum\frac{1}{a}=\frac{4s^2r^2}{4Rrs}\sum\frac{1}{a}=\frac{sr}{R}\sum\frac{1}{a}.$$

Equality holds if and only if the triangle is equilateral.

Left hand inequality

$$\sum\frac{h_a^2}{b^2+c^2}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{h_a^2}{b^2+c^2}}=3\sqrt[3]{\frac{\prod h_a^2}{\prod(b^2+c^2)}}\ge3\sqrt[3]{\frac{729r^6}{216R^6}}=3\cdot\frac{9r^2}{6R^2}=\frac{9r^2}{2R^2}$$

We have used above:

1. $\prod h_a^2\ge729r^6$, see $\prod h_a=\frac{2r^2s^2}{R}\overset{Cosnita\,\&\,Turtoiu}{\ge}27r^3$;

2. $\prod(b^2+c^2)\le216R^6$

see $\prod(b^2+c^2)=2[p^6+p^4(r^2-12Rr)+p^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3]\le$

$$\le 216R^6$$

Note: The right-hand inequality strengthens Problem JP.583 from RMM number 39, Winter 2025.

JP.583. In ΔABC:

$$\sum\frac{h_a^2}{b^2+c^2}\le\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Daniel Sitaru – Romania

Remark: We can write the inequalities

In ΔABC:

$$\frac{9r^2}{2R^2}\le\sum\frac{h_a^2}{b^2+c^2}\le\frac{F}{2R}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\le\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Remark: In the same way:

In ΔABC:

$$\frac{9r^2}{2R^2}\le\sum\frac{r_a^2}{b^2+c^2}\le\frac{8R^2-23r^2}{4Rr}$$

Marin Chirciu

Solution: **The right-hand inequality**

$$LHS=\sum\frac{r_a^2}{b^2+c^2}\overset{AM-GM}{\le}\sum\frac{r_a^2}{2bc}=\frac{1}{2}\sum\frac{r_a^2}{bc}=\frac{1}{2}\cdot\frac{8R^2+2Rr-p^2}{2Rr}=$$

$$=\frac{8R^2+2Rr-p^2}{4Rr}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{8R^2+2Rr-16Rr+5r^2}{4Rr}=\frac{8R^2-14Rr+5r^2}{4Rr}\overset{Euler}{\le}\frac{8R^2-23r^2}{4Rr}$$

We have used above:

$$\sum\frac{r_a^2}{bc}=\frac{8R^2+2Rr-p^2}{2Rr}$$

see $\sum\frac{r_a^2}{bc}=\sum\frac{\frac{F^2}{(s-a)^2}}{bc}=F^2\sum\frac{1}{bc(s-a)^2}=p^2r^2\frac{8R^2+2Rr-p^2}{2Rr^3p^2}=\frac{8R^2+2Rr-p^2}{2Rr}$

Equality holds if and only if the triangle is equilateral.

Left hand inequality

$$\sum\frac{r_a^2}{b^2+c^2}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{r_a^2}{b^2+c^2}}=3\sqrt[3]{\frac{\prod r_a^2}{\prod(b^2+c^2)}}\geq 3\sqrt[3]{\frac{729R^6}{216R^6}}=3\cdot\frac{9r^2}{6R^2}=\frac{9R^2}{2R^2}.$$

We have used above:

1. $\prod r_a^2\geq 729r^6$**, see** $\prod r_a=rp^2\overset{Mitrinovici}{\geq}27r^3$

2. $\prod(b^2+c^2)\leq 216R^6$

see $\prod(b^2+c^2)=2[p^6+p^4(r^2-12Rr)+p^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3]\leq$

$$\leq 216R^6$$

Solution 3 by proposer

Lemma 1: (BERGSTROM'S REVERSED INEQUALITY)

If $a,b,c,x,y,z>0$ **then:**

$$\frac{x^2}{b^2+c^2}+\frac{y^2}{c^2+a^2}+\frac{z^2}{a^2+b^2}\leq\frac{x^2+y^2}{4c^2}+\frac{y^2+z^2}{4a^2}+\frac{z^2+x^2}{4b^2}\quad (1)$$

Proof.

Inequality (1) can be written:

$$\frac{x^2+y^2}{4c^2}+\frac{y^2+z^2}{4a^2}+\frac{z^2+x^2}{4b^2}-\frac{x^2}{b^2+c^2}-\frac{y^2}{c^2+a^2}-\frac{z^2}{a^2+b^2}\geq 0$$

$$x^2\left(\frac{1}{4b^2}+\frac{1}{4c^2}-\frac{1}{b^2+c^2}\right)+y^2\left(\frac{1}{4a^2}+\frac{1}{4c^2}-\frac{1}{a^2+c^2}\right)+$$

$$+z^2\left(\frac{1}{4b^2}+\frac{1}{4a^2}-\frac{1}{a^2+b^2}\right)\geq 0$$

$$x^2\cdot\frac{(b^2+c^2)^2-4b^2c^2}{4b^2c^2(b^2+c^2)}+y^2\cdot\frac{(c^2+a^2)^2-4a^2c^2}{4a^2c^2(a^2+c^2)}+z^2\cdot\frac{(a^2+b^2)-4a^2b^2}{4a^2b^2(a^2+b^2)}\geq 0$$

$$x^2\cdot\frac{(b^2-c^2)^2}{4b^2c^2(b^2+c^2)}+y^2\cdot\frac{(c^2-a^2)^2}{4a^2c^2(a^2+c^2)}+z^2\cdot\frac{(a^2-b^2)^2}{4a^2b^2(a^2+b^2)}\geq 0$$

Equality holds for $a=b=c$**.**

Lemma 2: (SANTALO'S INEQUALITY)

If ΔABC **the following relationship holds:**

$$h_a\leq\sqrt{s(s-a)}\qquad (2)$$

Proof.

$$h_a=\frac{2F}{a}=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}=$$

$$=\sqrt{(s-b)(s-c)}\cdot\frac{2\sqrt{s(s-a)}}{a}\overset{AM-GM}{\leq}$$

$$\leq\frac{s-b+s-c}{2}\cdot\frac{2\sqrt{s(s-a)}}{a}=\frac{2s-b-c}{2}\cdot\frac{2\sqrt{s(s-a)}}{a}=$$

$$=\frac{(a+b+c-b-c)\sqrt{s(s-a)}}{a}=\frac{a\sqrt{s(s-a)}}{a}=\sqrt{s(s-a)}$$

Equality holds for $a=b=c$.

Back to the problem:

We take in (1): $x=h_a; y=h_b; z=h_c$

$$\sum_{cyc}\frac{h_a^2}{b^2+c^2}\overset{(1)}{\leq}\sum_{cyc}\frac{h_a^2+h_b^2}{4c^2}\overset{(2)}{\leq}\sum_{cyc}\frac{\left(\sqrt{s(s-a)}\right)^2+\left(\sqrt{s(s-b)}\right)^2}{4c^2}=$$

$$=\sum_{cyc}\frac{s(s-a)+s(s-b)}{4c^2}=\sum_{cyc}\frac{s(s-a+s-b)}{4c^2}=$$

$$=\frac{s}{4}\sum_{cyc}\frac{2s-a-b}{c^2}=\frac{s}{4}\sum_{cyc}\frac{a+b+c-a-b}{c^2}=\frac{s}{4}\sum_{cyc}\frac{1}{c}=\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Equality holds for $a=b=c$.

JP.584 Prove that in any triangle ABC the following inequality holds:

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{3}{16}\sum_{cyc}\sin A$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum_{cyc}\left(\cos^2\frac{A}{2}\left(\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\right)$$

$$=\sum_{cyc}\left(\frac{s(s-a)}{bc}\cdot\frac{s-a}{4R}\right)=\frac{s}{4R.4Rrs}\cdot\sum_{cyc}a(s-a)^2$$

$$=\frac{1}{16R^2r}\left(s^2\sum_{cyc}a-2s\sum_{cyc}a^2+\sum_{cyc}a^3\right)$$

$$=\frac{1}{16R^2r}\left(2s^3-4s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)\right)=\frac{s(2R-r)}{8R^2}$$

$$\overset{?}{\geq}\frac{3}{16}\sum_{cyc}\sin A=\frac{3s}{16R}\Leftrightarrow 4R-2r\overset{?}{\geq}3R\Leftrightarrow R\overset{?}{\geq}2r\rightarrow \text{true via Euler}$$

$$\therefore\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq\frac{3}{16}\sum_{cyc}\sin A\ \ \forall\Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$\sum\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum\left(\frac{1}{\sin\frac{A}{2}}\cos^3\frac{A}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=\frac{r}{4R}\sum\frac{\cot^3\frac{A}{2}}{\sin\frac{A}{2}}=\frac{r}{4R}\sum\left(\cot\frac{A}{2}\cos^2\frac{A}{2}\right)=\frac{r}{4R}\sum\cot\frac{A}{2}\left(1-\sin^2\frac{A}{2}\right)$$

$$=\frac{r}{4R}\left(\sum\cot\frac{A}{2}-\sum\cos\frac{A}{2}\sin\frac{A}{2}\right)=\frac{r}{4R}\left(\frac{s}{r}-\frac{1}{2}\sum\sin A\right)$$

$$=\frac{s}{4R}-\frac{1}{2}\cdot\frac{r}{4R}\cdot\sum\sin A\overset{\sum\sin A=\frac{s}{R}\ \&\ Euler}{\geq}\frac{1}{4}\sum\sin A-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\sum\sin A$$

$$=\frac{1}{4}\sum\sin A-\frac{1}{16}\sum\sin A=\frac{3}{16}\sum\sin A$$

Equality holds for an equilateral triangle.

Solution 3 by proposers

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum_{cyc}\cos\frac{A}{2}\left(1-\sin^2\frac{A}{2}\right)\sin\frac{B}{2}\sin\frac{C}{2}=$$

$$=\sum_{cyc}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\sum_{cyc}\cos\frac{A}{2}\sin^2\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=$$

$$=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{cyc}\sin\frac{A}{2}\cos\frac{A}{2}$$

But in any triangle holds $\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}=1$, therefore

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{cyc}\sin\frac{A}{2}\cos\frac{A}{2}$$

$$=\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-\frac{1}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sum_{cyc}\sin A$$

But $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, therefore

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum_{cyc}\sin A\left(\frac{1}{4}-\frac{1}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

But in any triangle holds $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$, so we obtain:

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge\sum_{cyc}\left(\frac{1}{4}-\frac{1}{16}\right)\sin A$$

Equality holds for an equilateral triangle.

JP.585 In $\Delta ABC, AA', BB', CC'$ are internal bisectors which intersect the circumcircle of triangle in the points A'', B'', C''. Prove that:

$$6\left(\left(\frac{R}{r}\right)^2-2\right)\le\frac{AA''}{A'A''}+\frac{BB''}{B'B''}+\frac{CC''}{C'C''}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

$$\frac{AA''}{A'A''}=\frac{AA'+A'AA''}{A'A''}=\frac{AA'}{A'A''}+1=\frac{AA''^2}{AA'\cdot A'A''}+1=\frac{w_a^2}{BA'\cdot A'C}+1\quad(1)$$

But, from Law of bisector: $BA'\cdot A'C=\frac{a^2bc}{(b+c)^2}$ (2)

From (1) and (2), it follows:

$$\frac{AA''}{A'A''}=\frac{4b^2c^2}{(b+c)^2}\cdot\cos\frac{A}{2}\cdot\frac{(b+c)^2}{a^2bc}+1=\frac{4bc}{a^2}\cdot\cos^2\frac{A}{2}=$$

$$=\frac{4s(s-a)}{a^2}+1=\frac{(b+c+a)(b+c-a)}{a^2}+1=$$

$$=\frac{(b+c)^2-a^2}{a^2}+1=\frac{(b+c)^2}{a^2}\text{ (and analogs)}$$

We must show that: $6\left(\left(\frac{R}{r}\right)^2-2\right)\le\sum_{cyc}\frac{(b+c)^2}{a^2}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\ge\frac{1}{3}\left(\sum_{cyc}\frac{b+c}{a}\right)^2=\frac{1}{3}\left(\frac{s^2+r^2-2Rr}{2Rr}\right)^2\ge$$

$$\overset{Gerretsen}{\ge}\frac{1}{3}\left(\frac{14Rr-4r^2}{2Rr}\right)^2=\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\overset{CBS}{\le}\sum_{cyc}\frac{2(b^2+c^2)}{a^2}=2\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)$$

But, $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{a^2}\le\frac{R^2}{r^2}-2$. Finally, we get:

$$\sum_{cyc}\frac{(b+c)^2}{a^2}\le 2\left(\frac{3R^2}{r^2}-6\right)=6\left(\frac{R^2}{r^2}-2\right)$$

PROBLEMS FOR SENIORS

SP.571 For given $n\ge 3$, prove that 3 is the largest positive value of the constant k such that:

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\ge k(a_1+a_2+\cdots+a_n-n)$$

for any $a_1\ge a_2\ge\cdots\ge a_{n-1}\ge 1\ge a_n>0$ with

$$a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1=n.$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Choosing $a_2=\cdots=a_{n-1}=1$, then inequality becomes $\frac{1}{a_1}+\frac{1}{a_n}-2\ge k(a_1+a_n-2)$, where $a_1\ge 1\ge a_n>0$ such that $a_1a_n+a_1+a_n=3$. Let $p=a_1a_n$. From $3=a_1a_n+a_1+a_n\ge p+2\sqrt{p}$, we get $p\in(0,1]$. Write the inequality as follows:

$$\frac{3-p}{p}-2\ge k(1-p),\ \ (1-p)(3-kp)\ge 0.$$

It is true if and only if $3-kp\geq 0$ for $p\in(0,1)$. From the necessary condition $\lim_{p\to 1}(3-kp)\geq 0$, we get $k\leq 3$. To show that 3 is the largest positive value of k, we need to prove the inequality

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+2n\geq 3(a_1+a_2+\cdots+a_n).$$

By the AM-HM inequality, we have $\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}\geq\frac{n-2}{S}$, where $S=\frac{a_2+\cdots+a_{n-1}}{n-2}\geq 1$. So, it suffices to show that $E\geq 0$, where

$$E=\frac{1}{a_1}+\frac{1}{a_n}+\frac{n-2}{S}+2n-3[a_1+a_n+(n-2)S].$$

By Lemma below, we have $(n-3)S^2+(a_1+a_n)S+a_1a_n<n$. Since the expression E decreases when a_1 increases, we may increase a_1 to have

$$(n-3)S^2+(a_1+a_n)S+a_1a_n=n.$$

Denoting $x=\frac{a_1+a_n}{2}$, we need to show that

$$\frac{2x}{n-(n-3)S^2-2Sx}+\frac{n-2}{S}+2n-3[2x+(n-2)S]\geq 0$$

for $(n-3)S^2+2Sx+a_1a_2=n$. From $(S-a_1)(S-a_n)\leq 0$, we obtain:

$$2Sx\geq a_1a_n+S^2=n-2Sx-(n-4)S^2$$

therefore

$$4Sx\geq n-(n-4)S^2.$$

For fixed S, the desired inequality is equivalent to $F(x)\geq 0$, where

$$F(x)=12S^2x^2+[6(2n-5)S^2-4nS-8n+6]Sx+$$
$$+[n-(n-3)S^2][n-2+2nS-3(n-2)S^2]$$

Since

$$F'(x)=24S^2x+6(2n-5)S^3-4nS^2-(8n-6)S\geq$$
$$\geq 6S[n-(n-4)S^2]+6(2n-5)S^3-4nS^2-(8n-6)S$$
$$=6(n-1)S^3-4nS^2-(2n-6)S\geq 6(n-1)S^2-4nS^2-(2n-6)S=$$
$$=n(n-3)S(S-1)\geq 0,$$

$F(x)$ is increasing, hence

$$F(x)\ge F\left(\frac{n-(n-4)S^2}{4S}\right)=\frac{n[3(n-2)S^4-4(n-2)S^3-2nS^2+4nS-n-2]}{4}$$

$$=\frac{n(S-1)^2[3(n-2)S^2+2(n-2)S-n-2]}{4}\ge 0.$$

The proof is completed. For $k=3$, the equality occurs when $a_1=a_2=\cdots=a_n=1$.

Lemma:

Let $n\ge 3$. If $a_1\ge a_2\ge\cdots\ge a_n\ge 0$ such that $a_1a_2+a_2a_3+\cdots+a_na_1=n$, then

$$(n-3)S^2+(a_1+a_n)S+a_1a_n\le n,$$

where $S=\frac{a_2+\cdots+a_{n-1}}{n-2}$.

Proof:

For $n=3$, the inequality is an equality. For $n\ge 4$, we write the desired inequality in the homogeneous form:

$$(n-3)S^2+(a_1+a_n)S+a_1a_n\le a_1a_2+a_2a_3+\cdots+a_na_1,$$

which is equivalent to

$$(n-3)S^2+a_1(S-a_2)+a_n(S-a_{n-1})\le a_2a_3+\cdots+a_{n-2}a_{n-1}.$$

Since $S-a_2\le 0$ and $S-a_{n-1}\ge 0$, it suffices to show that

$$(n-3)S^2+a_2(S-a_2)+a_{n-1}(S-a_{n-1})\le a_2a_3+\cdots+a_{n-2}a_{n-1},$$

which can be rewritten as

$$a_2a_3+\cdots+a_{n-2}a_{n-1}\ge(n-3)S^2+(a_2+a_{n-1})S-a_2^2-a_{n-1}^2.$$

Since the sequence $a_2,a_3,\dots,a_{n-2}$ and $a_3,a_4,\dots,a_{n-1}$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3+\cdots+a_{n-2}a_{n-1})\ge(a_2+\cdots+a_{n-2})(a_3+\cdots+a_{n-1})=$$

$$=\big((n-2)S-a_{n-1}\big)\big((n-2)S-a_2\big).$$

Thus, it suffices to show that

$$\frac{\big((n-2)S-a_{n-1}\big)\big((n-2)S-a)2\ \big)}{n-3}\ge(n-3)S^2+(a_2+a_{n-1})S-a_2^2-a_{n-1}^2,$$

which is equivalent to

$$(2n-5)S^2-(2n-5)(a_2+a_{n-1})S+(n-3)\big(a_2^2+a_{n-1}^2\big)+a_2a_{n-1}\ge 0,$$

$$(2n-5)(2S-a_2-a_{n-1})^2+(2n-7)(a_2-a_{n-1})^2\ge 0.$$

Clearly, the last inequality is true.

SP.572 **Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$ and $a^4bc \ge 1$, and let**

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Prove that:

$$F(a,b,c) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Since $F(a,b,c) \ge 0$ and $F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right) \ge 0$ (by the AM-GM inequality), it suffices to prove the homogeneous inequality

$$F(a,b,c) \ge (a^4bc)^{\frac{1}{3}} \cdot F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$$

for $a = \min\{a, b, c\}$. Due to homogeneity, we may set $a = 1$, hence $b, c \ge 1$. Thus, we need to show that

$$(bc)^{\frac{1}{3}} - \frac{3bc}{b+c+bc} \ge (bc)^{\frac{1}{3}}\left[\frac{1}{(bc)^{\frac{1}{3}}} - \frac{3}{1+b+c}\right].$$

Denote

$$s = \frac{b+c}{2}, \quad p = \sqrt{bc},$$

with $s \ge p \ge 1$. The desired inequality is equivalent to

$$p^{\frac{2}{3}} - \frac{3p^2}{2s+p^2} \ge 1 - \frac{3p^{\frac{2}{3}}}{2s+1},$$

$$p^{\frac{2}{3}} - 1 \ge 3p^{\frac{2}{3}}\left(\frac{p^{\frac{4}{3}}}{2s+p^2} - \frac{1}{2s+1}\right),$$

$$\frac{p^{\frac{2}{3}}}{3p^{\frac{2}{3}}} \geq \frac{2s\left(p^{\frac{4}{3}}-1\right)-p^{\frac{4}{3}}\left(p^{\frac{2}{3}}-1\right)}{(2s+p^2)(2s+1)}.$$

It is true if

$$\frac{1}{3p^{\frac{2}{3}}} \geq \frac{2s\left(p^{\frac{2}{3}}+1\right)-p^{\frac{4}{3}}}{(2s+p^2)(2s+1)},$$

i.e.

$$4s^2-2As+4p^2 \geq 0, \quad A=3p^{\frac{4}{3}}+3p^{\frac{2}{3}}-p^2-1.$$

For the nontrivial case $A \geq 0$, since

$$4(4s^2-2As+4p^2)=(4s-A)^2+16p^2-A^2 \geq 16p^2-A^2=(4p-A)(4p+A),$$

it suffices to show that $4p-A \geq 0$, which is equivalent to

$$p^2-3p^{\frac{4}{3}}+4p-3p^{\frac{2}{3}}+1 \geq 0.$$

Denoting $p=x^3$, we need to show that

$$x^6-3x^4+4x^3-3x^2+1 \geq 0,$$

that is

$$(x-1)^2(x^4+2x^3+2x+1) \geq 0.$$

The proof is completed. The equality occurs for $a=b=c \geq 1$.

Remark. The inequality $F(a,b,c) \leq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ is true in the particular case $a,b,c \geq 1$ (which involves $a^4bc \geq 1$).

SP.573 Let $\lambda \geq \frac{1}{2}$ fixed. If $a,b,c>0, a+b+c \leq 4$ then find the maximum value of

$$P=\frac{\sqrt{abc}}{(a+b)(b+c)}-\frac{\lambda}{c(a+b)}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

Lemma 1: If $a,b,c>0$ then:

$$\frac{\sqrt{abc}}{(a+b)(b+c)} \le \frac{1}{2\sqrt{a+b+c}}$$

Proof:

Using means inequality we obtain:

$$(a+b)(b+c) = b^2+ab+bc+ca = b(a+b+c)+ac \overset{AGM}{\geq} 2\sqrt{b(a+b+c)\cdot ac} =$$
$$= 2\sqrt{abc(a+b+c)}, \text{ with equality for } b(a+b+c) = ac.$$

Lemma 2: If $a,b,c>0$ then:

$$\frac{1}{c(a+b)} \geq \frac{4}{(a+b+c)^2}.$$

Proof:

With means inequality we obtain:

$$c+(a+b) \overset{AGM}{\geq} 2\sqrt{c(a+b)}, \text{ with equality for } c=a+b, \text{so } a+b+c \geq 2\sqrt{c(a+b)} \Rightarrow$$
$$\Rightarrow (a+b+c)^2 \geq 4c(a+b) \Rightarrow c(a+b) \le \frac{(a+b+c)^2}{4} \Rightarrow \frac{1}{c(a+b)} \geq \frac{4}{(a+b+c)^2}.$$

Let's get back to the main problem. Using the above Lemmas, we obtain:

$$P = \frac{\sqrt{abc}}{(a+b)(a+c)} - \frac{\lambda}{c(a+b)} \le \frac{1}{2\sqrt{a+b+c}} - \frac{4\lambda}{(a+b+c)^2}$$

Denoting $\sqrt{a+b+c} = t$ we have $0 < t \le 2$ and $P \le \frac{1}{2t} - \frac{4\lambda}{t^4}$.

Considering the function $f:(0,2] \to \mathbb{R}, f(t) = \frac{1}{2t} - \frac{4\lambda}{t^4}$.

We have $f'(t) = \frac{32\lambda - t^3}{t^5} > 0, 0 < t \le 2, \lambda \geq \frac{1}{2}$. It follows that the function f is strictly increasing on $(0,2]$, so $f(t) \le f(2) = \frac{1}{4} - \frac{\lambda}{9}$

It follows $P \le \frac{1}{2t} - \frac{4\lambda}{t^4} = f(t) \le f(2) = \frac{1-\lambda}{4}$.

From $P \le \frac{1}{4} - \frac{\lambda}{9}$, with equality for $a+b+c=4, b(a+b+c)=ac, c=a+b$, namely for $(a,b,c) = \left(\frac{4}{3}, \frac{2}{3}, 2\right)$, we deduce that max $P = \frac{1-\lambda}{4}$ and the maximum is attained for $(a,b,c) = \left(\frac{4}{3}, \frac{2}{3}, 2\right)$.

SP.574 If $a, b, c, d > 0; a^2+b^2+c^2+d^2=4$ then:

$$\frac{1}{(1+ab)^3}+\frac{1}{(1+ac)^3}+\frac{1}{(1+ad)^3}+\frac{1}{(1+bc)^3}+\frac{1}{(1+bd)^3}+\frac{1}{(1+cd)^3}\ge\frac{3}{4}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

We have :

$$a^2+b^2+c^2\ge ab+bc+ca\to(1)$$

$$a^2+b^2+d^2\ge ab+bd+da\to(2)$$

$$a^2+c^2+d^2\ge ac+cd+da\to(3)$$

$$b^2+c^2+d^2\ge bc+cd+bd\to(4)$$

and via $(1)+(2)+(3)+(4)$, we get :

$$3(a^2+b^2+c^2+d^2)\ge 2(ab+ac+ad+bc+bd+cd)$$
$$\overset{a^2+b^2+c^2+d^2=4}{\Rightarrow}\quad 12\ge 2(ab+ac+ad+bc+bd+cd)$$
$$\Rightarrow ab+ac+ad+bc+bd+cd\overset{(*)}{\le}6 \text{ and via Radon,}$$
$$\frac{1}{(1+ab)^3}+\frac{1}{(1+ac)^3}+\frac{1}{(1+ad)^3}+\frac{1}{(1+bc)^3}+\frac{1}{(1+bd)^3}+\frac{1}{(1+cd)^3}=$$
$$=\frac{1^4}{(1+ab)^3}+\frac{1^4}{(1+ac)^3}+\frac{1^4}{(1+ad)^3}+\frac{1^4}{(1+bc)^3}+\frac{1^4}{(1+bd)^3}+\frac{1^4}{(1+cd)^3}\ge$$
$$\frac{(1+1+1+1+1+1)^4}{(6+ab+ac+ad+bc+bd+cd)^3}\overset{\text{via }(*)}{\ge}\frac{6^4}{(6+6)^3}=6.\left(\frac{6}{12}\right)^3=\frac{3}{4}$$

$$\therefore\frac{1}{(1+ab)^3}+\frac{1}{(1+ac)^3}+\frac{1}{(1+ad)^3}+\frac{1}{(1+bc)^3}+\frac{1}{(1+bd)^3}+\frac{1}{(1+cd)^3}\ge\frac{3}{4}$$

$$\forall\, a,b,c,d>0\,\Big|\,a^2+b^2+c^2+d^2=4,''='' \text{ iff } a=b=c=d=1\text{ (QED)}$$

Solution 2 by proposer

$$(a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2\ge 0$$
$$a^2-2ab+b^2+a^2-2ac+c^2+a^2-2ad+d^2+b^2-2bc+c^2+$$
$$+b^2-2bd+d^2+c^2-2cd+d^2\ge 0$$

$$3(a^2+b^2+c^2+d^2)\ge 2(ab+ac+ad+bc+bd+cd)$$

$$ab+ac+ad+bc+bd+cd\le\frac{3}{2}(a^2+b^2+c^2+d^2)=\frac{3\cdot 4}{2}=6$$

$$ab+ac+ad+bc+bd+cd\le 6\quad (1)$$

$$\frac{1}{(1+ab)^3}+\frac{1}{(1+ac)^3}+\frac{1}{(1+ad)^3}+\frac{1}{(1+bc)^3}+\frac{1}{(1+bd)^3}+\frac{1}{(1+cd)^3}=$$

$$=\frac{1^4}{(1+ab)^3}+\frac{1^4}{(1+ac)^3}+\frac{1^4}{(1+ad)^3}+\frac{1^4}{(1+bc)^3}+\frac{1^4}{(1+bd)^3}+\frac{1^4}{(1+cd)^3}\ge$$

$$\overset{RADON}{\ge}\frac{(1+1+1+1+1+1)^4}{(6+ab+ac+ad+bc+bd+cd)^3}\overset{(1)}{\ge}$$

$$\ge\frac{6^4}{(6+6)^3}=\frac{6^4}{8\cdot 6^3}=\frac{6}{8}=\frac{3}{4}$$

Equality holds for $a=b=c=d=1$.

SP.575 If $x,y,z>0$, then prove that:

$$\frac{\sum x^2-\sum xy}{8(\sum x)^2}+\left(\sum x\right)\sum\frac{1}{2x+y+z}\le\frac{3(\sum x)^2}{4\sum xy}$$

Proposed by Neculai Stanciu – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $\mathrm{y}+\mathrm{z}=a,\mathrm{z}+x=\mathrm{b},x+\mathrm{y}=\mathrm{c}\Rightarrow a+\mathrm{b}-\mathrm{c}=2\mathrm{z}>0,$
$\mathrm{b}+\mathrm{c}-a=2x>0$ *and* $\mathrm{c}+a-\mathrm{b}=2\mathrm{y}>0\Rightarrow a+\mathrm{b}>c,\mathrm{b}+\mathrm{c}>a,\mathrm{c}+a>b$
$\Rightarrow a,\mathrm{b},\mathrm{c}$ **form sides of** a **triangle with semiperimeter, circumradius and inradius**
$=\mathrm{s},\mathrm{R},\mathrm{r}\ (say)\Rightarrow\sum_{cyc}x=\mathrm{s}\to(1),\sum_{cyc}xy=4\mathrm{Rr}+\mathrm{r}^2\to(2),$

$$\sum_{cyc}x^2=\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2\to(3)\ \text{and now,}$$

$$\frac{\sum_{cyc}x^2-\sum_{cyc}xy}{8\left(\sum_{cyc}x\right)^2}+\left(\sum_{cyc}x\right)\sum_{cyc}\frac{1}{2x+\mathrm{y}+\mathrm{z}}\le\frac{3\left(\sum_{cyc}x\right)^2}{4\sum_{cyc}xy}$$

$$\Leftrightarrow\frac{\sum_{cyc}x^2-\sum_{cyc}xy}{8\left(\sum_{cyc}x\right)^2}+\left(\sum_{cyc}x\right)\sum_{cyc}\frac{1}{(x+\mathrm{y})+(\mathrm{z}+x)}\le\frac{3\left(\sum_{cyc}x\right)^2}{4\sum_{cyc}xy}$$

$$\Leftrightarrow\frac{\sum_{cyc}x^2-\sum_{cyc}xy}{8\left(\sum_{cyc}x\right)^2}+\left(\sum_{cyc}x\right)\sum_{cyc}\frac{1}{\mathrm{c}+\mathrm{b}}\le\frac{3\left(\sum_{cyc}x\right)^2}{4\sum_{cyc}xy}$$

$$\Leftrightarrow \frac{\sum_{cyc} x^2 - \sum_{cyc} xy}{8\left(\sum_{cyc} x\right)^2} + \left(\sum_{cyc} x\right) \cdot \frac{3\sum_{cyc} ab + \sum_{cyc} a^2}{2s(s^2+2Rr+r^2)} \le \frac{3\left(\sum_{cyc} x\right)^2}{4\sum_{cyc} xy}$$

$$\overset{\text{via }(1),(2)\text{ and }(3)}{\Leftrightarrow} \frac{s^2-12Rr-3r^2}{8s^2} + \frac{4s^2+s^2+4Rr+r^2}{2(s^2+2Rr+r^2)} \le \frac{3s^2}{4(4Rr+r^2)}$$

$$\Leftrightarrow 6s^4(s^2+2Rr+r^2) -$$

$$(4Rr+r^2)\left(4s^2(5s^2+4Rr+r^2)+(s^2-12Rr-3r^2)(s^2+2Rr+r^2)\right) \boxed{\overset{(*)}{\ge}} 0$$

$$\text{and } \because P = 6(s^2-16Rr+5r^2)^3 + (216Rr-105r^2)(s^2-16Rr+5r^2)^2 +$$

$$2r^2(1140R^2-1327Rr+299r^2)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\ge} 0$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow 1464t^3 - 3842t^2 + 1967t - 278 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1464t^2-914t+139) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\sum_{cyc} x^2 - \sum_{cyc} xy}{8\left(\sum_{cyc} x\right)^2} + \left(\sum_{cyc} x\right)\sum_{cyc} \frac{1}{2x+y+z} \le \frac{3\left(\sum_{cyc} x\right)^2}{4\sum_{cyc} xy}\ \forall x,y,z>0,$$

$$'' ='' \text{ iff } x=y=z \text{ (QED)}$$

Solution 2 by proposer

$$2\sum a^2 - 2\sum ab = \sum(a-b)^2 \text{ and}$$

$$\left(\sum a\right)\left(\sum \frac{1}{a}\right) - 9 = \sum \frac{(a-b)^2}{ab}.$$

Because $4(\sum x) = \sum(2x+y+z)$ the given inequality becomes successively

$$\frac{\sum x^2 - \sum xy}{2(\sum x)^2} + ((2x+y+z))\left(\sum \frac{1}{2x+y+z}\right) - 9 \le \frac{3(\sum x)^2}{\sum xy} - 9$$

$$\Leftrightarrow \frac{\sum x^2 - \sum xy}{2(\sum x)^2} + \sum \frac{(2x+y+z-x-2y+z)^2}{(2x+y+z)(x+2y+z)} \le \frac{3(\sum x^2 - \sum xy)}{\sum xy}$$

$$\Leftrightarrow \frac{\sum(x-y)^2}{4(\sum x)^2} + \sum \frac{(x-y)^2}{(2x+y+z)(x+2y+z)} \le \frac{3\sum(x-y)^2}{2\sum xy}.$$

It suffices to show that

$$\frac{(x-y)^2}{4(\sum x)^2} + \frac{(x-y)^2}{(2x+y+z)(x+2y+z)} \le \frac{3(x-y)^2}{2\sum xy}.$$

If $x=y$, we have equality; if $x \ne y$ we must to show that

$\frac{1}{(2x+y+z)(x+2y+z)} \le \frac{3}{2\sum xy} - \frac{1}{4(\sum x)^2}$, **and because**

$(2x+y+z)(x+2y+z) \geq (x+y+z)^2 \geq 3\sum xy$, it suffices to prove that

$$\frac{1}{3\sum xy} \leq \frac{3}{2\sum xy} - \frac{1}{4(\sum x)^2} \leq 4\left(\sum x\right)^2 \leq 18\left(\sum x\right)^2 - 3\sum xy \Leftrightarrow$$

$\Leftrightarrow 14(\sum x)^2 \geq 3\sum xy$, true. The proof is complete.

SP.576 If $a,b,c \geq 1$, then:

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} - \frac{a+b+c}{3} \geq \sqrt[3]{\frac{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}{3}} - \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Without loss of generality, we may assume that $c \geq b \geq a \geq 1$.

We write the inequality in the form $E(a,b,c) \geq 0$, where:

$$E(a,b,c) = \sqrt[3]{9(a^3+b^3+c^3)} - (a+b+c) - \sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)} + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

We shall prove that: $E(a,b,c) \geq E(a,x,x) \geq 0$, where:

$$E(a,x,x) = \sqrt[3]{9(a^3+2x^3)} - (a+2x) - \sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)} + \left(\frac{1}{a}+\frac{2}{x}\right)$$

and $x = \sqrt{bc} \geq a$;

a. We will prove the inequality: $E(a,b,c) \geq E(a,x,x)$.

The inequality can be written as follows:

$A - B \geq C - D$, where: $A = \sqrt[3]{9(a^3+b^3+c^3)} - \sqrt[3]{9(a^3+2x^3)}$;

$$A = \frac{9\left(b\sqrt{b} - c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(a^3+b^3+c^3)}\right)^2 + \sqrt[3]{9(a^3+b^3+c^3)}\sqrt[3]{9(a^3+2x^3)} + \left(\sqrt[3]{9(a^3+2x^3)}\right)^2}$$

$$A \geq \frac{9\left(b\sqrt{b} - c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2 + \sqrt[3]{9(x^3+b^3+c^3)}3x + 9x^2}$$

$$B = a+b+c-(a+2x) = b+c-2x = \left(\sqrt{b} - \sqrt{c}\right)^2$$

$$C=\sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}-\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}$$

$$C=\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^6\left(\left(\sqrt[3]{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}\right)^2+\sqrt[3]{9\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}+\left(\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\right)^2\right)}$$

$$C\le\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^6\left(\left(\sqrt[3]{9\left(\frac{1}{x^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\right)^2+\sqrt[3]{9\left(\frac{1}{x^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)}\frac{3}{x}+\frac{9}{x^2}\right)}$$

$$C\le\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}$$

$$D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\left(\frac{1}{a}+\frac{2}{x}\right)=\frac{1}{b}+\frac{1}{c}-\frac{2}{x}=\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

To prove the inequality $E(a,b,c)\ge E(a,x,x)$ it is enough to prove that:

$$\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}-\left(\sqrt{b}-\sqrt{c}\right)^2\ge$$

$$\frac{9\left(b\sqrt{b}-c\sqrt{c}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}-\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

Or:

$$\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{9\left(b+c+\sqrt{bc}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}-1\right)\ge$$

$$\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{9\left(b+c+\sqrt{bc}\right)^2}{x^2\left(\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2\right)}-\frac{1}{x^2}\right)$$

For $b=c$ this relationship it is true (case of equality). For $b\ne c$ we will show that:

$$\frac{9\left(b+c+\sqrt{bc}\right)^2}{\left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2+\sqrt[3]{9(x^3+b^3+c^3)}3x+9x^2}\left(1-\frac{1}{x^2}\right)\ge1-\frac{1}{x^2}$$

$$9\left(b+c+\sqrt{bc}\right)^2 \geq \left(\sqrt[3]{9(x^3+b^3+c^3)}\right)^2 + \sqrt[3]{9(x^3+b^3+c^3)}3x + 9x^2$$

$$9\left(b+c+\sqrt{bc}\right)^2 \geq \left(\sqrt[3]{9\left(bc\sqrt{bc}+b^3+c^3\right)}\right)^2 + \sqrt[3]{9\left(bc\sqrt{bc}+b^3+c^3\right)}3\sqrt{bc} + 9bc$$

Let us denote: $y = \frac{b}{c}; 0 < y \leq 1$

The previous inequality is written in the equivalent form:

$$9\left(y+1+\sqrt{y}\right)^2 \geq \left(\sqrt[3]{9\left(y\sqrt{y}+y^3+1\right)}\right)^2 + \sqrt[3]{9\left(y\sqrt{y}+y^3+1\right)}3\sqrt{y} + 9y$$

But, we have for any $y, 0 < y \leq 1$: $9\left(y\sqrt{y}+y^3+1\right) \leq \left(\frac{9}{10}y+\frac{21}{10}\right)^3$

Indeed, we calculate:

$$\left(\frac{9}{10}y+\frac{21}{10}\right)^3 - 9\left(y\sqrt{y}+y^3+1\right) =$$

$$= \frac{9}{1000}\left(1-\sqrt{y}\right)\left(1352y+919y^2+29\sqrt{y}+352y\sqrt{y}+919y^2\sqrt{y}+29\right) \geq 0$$

Returning to the main inequality, it suffices to prove:

$$9\left(y+1+\sqrt{y}\right)^2 \geq \left(\frac{9}{10}y+\frac{21}{10}\right)^2 + \left(\frac{9}{10}y+\frac{21}{10}\right)3\sqrt{y} + 9y$$

$$100\left(y+1+\sqrt{y}\right)^2 - (3y+7)^2 - 10(3y+7)\sqrt{y} - 100y =$$

$$= 158y + 91y^2 + 130\sqrt{y} + 170y\sqrt{y} + 51 \geq 0$$

Then, the inequality $E(a,b,c) \geq E(a,x,x)$ is proved.

b. We will prove the inequality: $E(a,x,x) \geq 0$. We have to show that:

$$\sqrt[3]{9(a^3+2x^3)} - (a+2x) \geq \sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)} - \left(\frac{1}{a}+\frac{2}{x}\right)$$

The inequality is successively written in the following equivalent forms:

$$\frac{9(a^3+2x^3)-(a+2x)^3}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2 + \sqrt[3]{9(a^3+2x^3)}(a+2x) + (a+2x)^2} \geq$$

$$\geq \frac{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)-\left(\frac{1}{a}+\frac{2}{x}\right)^3}{\left(\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\right)^2+\sqrt[3]{9\left(\frac{1}{a^3}+\frac{2}{x^3}\right)}\left(\frac{1}{a}+\frac{2}{x}\right)+\left(\frac{1}{a}+\frac{2}{x}\right)^2}$$

$$\frac{2(4a+5x)(a-x)^2}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2+\sqrt[3]{9(a^3+2x^3)}(a+2x)+(a+2x)^2}\geq$$

$$\geq \frac{\frac{1}{a^3x^3}2(5a+4x)(a-x)^2}{\frac{1}{a^2x^2}\left(\left(\sqrt[3]{9(x^3+2a^3)}\right)^2+\sqrt[3]{9(x^3+2a^3)}(x+2a)+(x+2a)^2\right)}$$

$$\frac{2(4a+5x)(a-x)^2}{\left(\sqrt[3]{9(a^3+2x^3)}\right)^2+\sqrt[3]{9(a^3+2x^3)}(a+2x)+(a+2x)^2}\geq$$

$$\geq \frac{2(5a+4x)(a-x)^2}{ax\left(\sqrt[3]{9(x^3+2a^3)}\right)^2+ax\sqrt[3]{9(x^3+2a^3)}(x+2a)+ax(x+2a)^2}$$

The last inequality holds because:

a. $4a+5x\geq 5a+4x$**, equivalent to:** $x\geq a$.

b. $ax\left(\sqrt[3]{9(x^3+2a^3)}\right)^2\geq\left(\sqrt[3]{9(a^3+2x^3)}\right)^2$**, equivalent to:**

$$a^3(x^9-a^3)+4x^6(a^6-1)+4a^3x^3(a^6-1)\geq 0$$

c. $ax\sqrt[3]{9(x^3+2a^3)}(x+2a)\geq\sqrt[3]{9(a^3+2x^3)}(a+2x)$**, results from inequalities b and d**

d. $ax(x+2a)^2\geq(a+2x)^2$**, equivalent to:**

$$a(x^3-a)+4x^2(a^2-1)+4ax(a^2-1)\geq 0$$

Thus, the inequality $E(a,x,x)\geq 0$ **is proved.**

So, the inequality in the statement: $E(a,b,c)\geq 0$ **is also proved.**

SP.577 Find all $n\in\mathbb{N}^*$ **such that:**

$$\int_0^1(\sin x)^{2n-2}\cdot(\cos x)^{2n}dx\geq\frac{1}{4^{1011}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\frac{1}{4^{1011}} \leq \int_0^1 (\sin x)^{2n-2} \cdot (\cos x)^{2n} dx =$$

$$= \int_0^1 (\sin^2 x)^{n-1} \cdot (\cos^2 x)^n dx = \int_0^1 \cos^2 x \cdot (\sin^2 x \cdot \cos^2 x)^{n-1} dx =$$

$$= \int_0^1 \cos^2 x \cdot \left(\sin^2 x\,(1 - \sin^2 x)\right)^{n-1} dx \overset{AM-GM}{\leq}$$

$$\leq \int_0^1 \cos^2 x \cdot \left(\left(\frac{\sin^2 x + 1 - \sin^2 x}{2}\right)^2\right)^{n-1} dx =$$

$$= \int_0^1 \cos^2 x \cdot \frac{1}{2^{2(n-1)}} dx = \frac{1}{4^{n-1}} \int_0^1 \cos^2 x\, dx < \frac{1}{4^{n-1}} \cdot \int_0^1 dx = \frac{1}{4^{n-1}}$$

$$\frac{1}{4^{1011}} < \frac{1}{4^{n-1}} \Rightarrow 4^{n-1} < 4^{1011} \Rightarrow n - 1 < 1011$$

$$\Rightarrow n < 1012 \Rightarrow n \in \{1, 2, 3, \dots, 1011\}$$

SP.578 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\log^2_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log^2_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) \geq 2$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\sqrt{\frac{\log^b_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log^2_{\cos x}\left(\frac{\sin x}{\sin x + \cos x}\right)}{2}} \geq$$

$$\overset{QM-HM}{\geq} \frac{2}{\frac{1}{\log_{\sin x}\left(\frac{\sin x}{\sin x + \cos x}\right)} + \frac{1}{\log_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right)}} =$$

$$= \frac{2}{\log_{\left(\frac{\sin 2x}{\sin x + \cos x}\right)} \sin x + \frac{1}{\log_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right)}} =$$

$$= \frac{2}{\log_{\left(\frac{\sin 2x}{\sin x + \cos x}\right)} \sin x + \log_{\left(\frac{\sin 2x}{\sin x + \cos x}\right)} \cos x} =$$

$$=\frac{2}{\log_{\left(\frac{\sin 2x}{\sin x+\cos x}\right)}(\sin x+\cos x)}=2\log_{\sin x+\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)\geq 1$$

$$\Leftrightarrow \log_{\sin x\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)^2\geq \log_{\sin x\cos x}(\sin x\cos x)$$

$$\Leftrightarrow \left(\frac{\sin 2x}{\sin x+\cos x}\right)^2\leq \sin x\cos x$$

$$\Leftrightarrow 4\sin^2 x\cos^2 x\leq \sin x\cos x\,(\sin x+\cos x)^2$$

$$\Leftrightarrow 4\sin x\cos x\leq (\sin x+\cos x)^2\Leftrightarrow 4\sin x\cos x\leq \sin^2 x+\cos^2 x+2\sin x\cos x$$

$$\Leftrightarrow \sin^2 x+\cos^2 x-2\sin x\cos x\geq 0\Leftrightarrow (\sin x-\cos x)^2\geq 0$$

Equality holds for $x=\frac{\pi}{4}\Leftrightarrow \sin x=\cos x$

$$\sqrt{\frac{\log^2_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log^2_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)}{2}}\geq 1$$

$$\frac{\log^2_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log^2_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)}{2}\geq 1$$

$$\log^2_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log^2_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)\geq 2$$

SP.579 If $f:[a,b]\to\mathbb{R};\ 0<a\leq b;\ f$ – continuous then:

$$\frac{b^{4047}-a^{4047}}{4047}+\int_a^b f^2(x^{2024})\,dx\geq \frac{1}{1012}\int_{a^{2024}}^{b^{2024}} f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$\left(x^{2023}-f(x^{2024})\right)^2\geq 0\Rightarrow \int_a^b\left(x^{2023}-f(x^{2024})\right)^2dx\geq 0$$

$$\int_a^b x^{4046}\,dx+\int_a^b f^2(x^{2024})\,dx-2\int_a^b x^{2023}f(x^{2024})dx\geq 0\qquad (1)$$

For the integral $\int_a^b x^{2023}f(x^{2024})dx$ denote:

$$y=x^{2024}\Rightarrow dy=2024x^{2023}dx$$

If $x=a\Rightarrow y=a^{2024}$

If $x = b \Rightarrow y = b^{2024}$

$$\int_a^b x^{2023} f(x^{2024})dx = \int_{a^{2024}}^{b^{2024}} f(y) \cdot \frac{1}{2024} dy = \frac{1}{2024} \int_{a^{2024}}^{b^{2024}} f(x)\,dx \quad (2)$$

Replacing (2) in (1):

$$\int_a^b x^{4047}\,dx + \int_a^b f^2(x^{2024})\,dx - 2 \cdot \frac{1}{2024} \int_{a^{2024}}^{b^{2024}} f(x)\,dx \geq 0$$

$$\frac{b^{4048} - a^{4048}}{4048} + \int_a^b f^2(x^{2024})\,dx \geq \frac{1}{1012} \int_{a^{2024}}^{b^{2024}} f(x)\,dx$$

Equality holds for $a = b$.

SP.580 If $a \in \mathbb{R}; m, n \geq 1$ then:

$$(2 + \cos a)^{m+n} + (3 + \cos a)^m + (3 + \cos a)^n \leq (3 + \cos a)^{m+n} + 1$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Lemma 1: If $x > 1; p \geq 1$ then:

$$(x - 1)^p \leq x^p - 1 \quad (1)$$

Proof.

Let be $f: (1, \infty) \to \mathbb{R}; f(x) = (x - 1)^p - x^p + 1$

$$f'(x) = p(x - 1)^{p-1} - px^{p-1} = p((x - 1)^{p-1} - x^{p-1}) < 0$$

because $x - 1 < x; (\forall) x > 1$

f decreasing on $(1, \infty)$

$$\sup_{x>1} f(x) = \lim_{\substack{x\to 1\\ x>1}} f(x) = \lim_{\substack{x\to 1\\ x>1}} ((x - 1)^p - x^p + 1) =$$

$$= (1 - 1)^p - 1^p + 1 = -1 + 1 = 0$$

$$f(x) \leq 0; (\forall) x > 1 \Rightarrow (x - 1)^p - x^p + 1 \leq 0$$

$$(x - 1)^p \leq x^p - 1$$

Equality holds for $p = 1$.

Lemma 2: If $x > 1; m, n \geq 1$ then:

$$(x - 1)^{m+n} + x^m + x^n \leq x^{m+n} + 1 \quad (2)$$

By (1):

$$(x-1)^m \leq x^m - 1 \quad (3)$$
$$(x-1)^n \leq x^n - 1 \quad (4)$$

By multiplying (3); (4):

$$(x-1)^m \cdot (x-1)^n \leq (x^m-1)(x^n-1)$$
$$(x-1)^{m+n} \leq x^{m+n} - x^m - x^n + 1$$
$$(x-1)^{m+n} + x^m + x^n \leq x^{m+n} + 1$$

Back to the problem:

We take $x = 3 + \cos a > 1$ in (2):

$$(3+\cos a - 1)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$
$$(2+\cos a)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$

Equality holds for $m = n = 1$:

$$(2+\cos a)^2 + (3+\cos a)^1 + (3+\cos a)^1 = (3+\cos a)^2 + 1$$
$$4 + 4\cos a + \cos^2 a + 6 + 2\cos a = 9 + \cos^2 a + 6\cos a + 1$$
$$\cos^2 a + 6\cos a + 10 = \cos^2 a + 6\cos a + 10$$

SP.581 If $x, y, z \geq 1$ then:

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)\left(\frac{1}{x}+\frac{2}{y}+xy^2\right) \geq \left(x+y+z+\frac{1}{xyz}\right)\left(x+2y+\frac{1}{xy^2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)\left(\frac{1}{x}+\frac{2}{y}+xy^2\right) \geq \left(x+y+z+\frac{1}{xyz}\right)\left(x+2y+\frac{1}{xy^2}\right)$$

$$\Leftrightarrow \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz}{x+y+z+\frac{1}{xyz}} - 1 \geq \frac{x+2y+\frac{1}{xy^2}}{\frac{1}{x}+\frac{2}{y}+xy^2} - 1 \Leftrightarrow \frac{x^2y^2z^2 - 1 + \sum_{\text{cyc}} xy - xyz\left(\sum_{\text{cyc}} x\right)}{xyz\left(x+y+z+\frac{1}{xyz}\right)}$$

$$\geq \frac{(x^2y^2 - x^2y^4) + (1-y^2) - (2xy - 2xy^3)}{xy^2\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)}$$

$$\Leftrightarrow \frac{\prod_{\text{cyc}}(a+1)^2 - 1 + \sum_{\text{cyc}}\left((a+1)(b+1)\right) - \prod_{\text{cyc}}(a+1).\left(\sum_{\text{cyc}}(a+1)\right)}{xyz\left(x+y+z+\frac{1}{xyz}\right)} \geq$$

$$\frac{(x^2y^2-x^2y^4)+(1-y^2)-(2xy-2xy^3)}{xy^2\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)}\ (x=a+1,y=b+1,z=c+1)$$

$$\Leftrightarrow\frac{a^2b^2c^2+2abc\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2+abc\sum_{cyc}a+\sum_{cyc}a^2b+\sum_{cyc}ab^2+2abc}{xyz\left(x+y+z+\frac{1}{xyz}\right)}$$

$$\geq\frac{(1-y^2)(x^2y^2-2xy+1)}{xy^2\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)}$$

$$\Leftrightarrow\frac{a^2b^2c^2+2abc\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2+abc\sum_{cyc}a+\sum_{cyc}a^2b+\sum_{cyc}ab^2+2abc}{xyz\left(x+y+z+\frac{1}{xyz}\right)}$$

$$\overset{(*)}{\geq}\frac{-(y^2-1)(xy-1)^2}{xy^2\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)}\to\text{true}\because x,y,z\geq 1\Rightarrow a,b,c\geq 0\Rightarrow\text{LHS of }(*)\geq 0$$

$$\geq\frac{-(y^2-1)(xy-1)^2}{xy^2\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)}\ (\because -(y^2-1)\leq 0)\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)\geq\left(x+y+z+\frac{1}{xyz}\right)\left(x+2y+\frac{1}{xy^2}\right)$$

$$\forall\, x,y,z\geq 1,''=''\text{ iff } x=y=1\ (\text{QED})$$

Solution 2 by proposer

If $x,y\geq 1\Rightarrow xy\geq 1\Rightarrow xy-1\geq 0;x-1\geq 0;y-1\geq 0$

We will prove that:

$$\frac{1}{x}+\frac{1}{y}+xy\geq x+y+\frac{1}{xy}\quad (1)$$

$$y+x+x^2y^2\geq x^2y+xy^2+1$$

$$x^2y^2-1+x-x^2y+y-xy^2\geq 0$$

$$(xy-1)(xy+1)-x(xy-1)-y(xy-1)\geq 0$$

$$(xy-1)(xy+1-x-y)\geq 0$$

$$(xy-1)\big(x(y-1)-(y-1)\big)\geq 0$$

$$(xy-1)(y-1)(x-1)\geq 0\quad (\text{True})$$

We will prove that:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\geq x+y+z+\frac{1}{xyz}\qquad (2)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz=\frac{1}{x}+\frac{1}{y}+xy-xy+\frac{1}{z}+xyz\ge$$

$$\overset{(1)}{\ge} x+y+\frac{1}{xy}-xy+\frac{1}{z}+xyz$$

Remains to prove that:

$$x+y+\frac{1}{xy}-xy+\frac{1}{z}+xyz\ge x+y+z+\frac{1}{xyz}$$

$$\frac{1}{xy}-xy+\frac{1}{z}+xyz-z-\frac{1}{xyz}\ge 0$$

$$\frac{1}{xy}\left(1-\frac{1}{z}\right)+xy(z-1)-\frac{z^2-1}{z}\ge 0$$

$$\frac{1}{xyz}\cdot(z-1)+xy(z-1)-\frac{(z-1)(z+1)}{z}\ge 0$$

$$(z-1)\left(\frac{1}{xyz}+xy-\frac{z+1}{z}\right)\ge 0$$

$$(z-1)\left(\frac{1}{xyz}-\frac{1}{z}+xy-1\right)\ge 0$$

$$(z-1)\left(\frac{1-xy}{xyz}+xy-1\right)\ge 0$$

$$(z-1)\big(xyz(xy-1)-(xy-1)\big)\ge 0$$

$$(z-1)(xy-1)(xyz-1)\ge 0\quad \textbf{(True)}$$

Let $y=z$ in (2):

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{y}+xyy\ge x+y+y+\frac{1}{xyy}$$

$$\frac{1}{x}+\frac{2}{y}+xy^2\ge x+2y+\frac{1}{xy^2}\qquad (3)$$

By multiplying (2); (3):

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)\left(\frac{1}{x}+\frac{2}{y}+xy^2\right)\ge\left(x+y+z+\frac{1}{xyz}\right)\left(x+2y+\frac{1}{xy^2}\right)$$

Equality holds for $x=y=z=1$.

SP.582 If $0 < a \le b$ then:

$$\int_a^b (\sin x)^{2\sin^2 x} \cdot (1-\sin^2 x)^{1-\sin^2 x} dx \ge \frac{b-a}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$(\sin x)^{2\sin^2 x}(1-\sin^2 x)^{1-\sin^2 x} = (\sin^2 x)^{\sin^2 x}(\cos^2 x)^{\cos^2 x} \overset{GM-HM}{\ge}$$

$$\ge \left(\frac{\sin^2 x + \cos^2 x}{\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\cos^2 x}}\right)^{\sin^2 x + \cos^2 x} = \frac{1}{2}\ (1)$$

$$\int_a^b (\sin x)^{2\sin^2 x}(1-\sin^2 x)^{1-\sin^2 x}\, dx \overset{(1)}{\ge} \int_a^b \left(\frac{1}{2}\right) dx = \frac{b-a}{2}$$

Equality holds for $a = b$.

Solution 2 by proposer

By weighted AM-GM:

$$a_1^{b_1} \cdot a_2^{b_2} \le \frac{a_1 b_1 + a_2 b_2}{b_1 + b_2}; a_1, a_2, b_1, b_2 > 0$$

For: $a_1 = \frac{1}{\sin^2 x}; a_2 = \frac{1}{\cos^2 x}; b_1 = \sin^2 x; b_2 = \cos^2 x$

$$\left(\frac{1}{\sin^2 x}\right)^{\sin^2 x} \cdot \left(\frac{1}{\cos^2 x}\right)^{\cos^2 x} \le \frac{\frac{1}{\sin^2 x} \cdot \sin^2 x + \frac{1}{\cos^2 x} \cdot \cos^2 x}{\sin^2 x + \cos^2 x}$$

$$\frac{1}{(\sin^2 x)^{\sin^2 x}} \cdot \frac{1}{(\cos^2 x)^{\cos^2 x}} \le \frac{1+1}{1} = 2$$

$$\frac{1}{(\sin x)^{2\sin^2 x} \cdot (1-\sin^2 x)^{1-\sin^2 x}} \le 2$$

$$(\sin x)^{2\sin^2 x} \cdot (1-\sin^2 x)^{1-\sin^2 x} \ge \frac{1}{2}$$

$$\int_a^b (\sin x)^{2\sin^2 x} \cdot (1-\sin^2 x)^{1-\sin^2 x} dx \ge \int_a^b \frac{1}{2} dx = \frac{b-a}{2}$$

Equality holds for $a = b$.

SP.583 Prove that if $x,y,z\ge 1$ then:

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+xy+yz+zx\ge 2(x+y+z)+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Desired inequality}\Leftrightarrow\frac{2\left(\sum_{cyc}xy-xyz\sum_{cyc}x\right)}{xyz}\ge\frac{\sum_{cyc}x-xyz\sum_{cyc}xy}{xyz}$$

$$\Leftrightarrow\left(\sum_{cyc}xy\right)(2+xyz)\ge\left(\sum_{cyc}x\right)(1+2xyz)\Leftrightarrow\frac{\sum_{cyc}xy}{\sum_{cyc}x}-1\ge\frac{1+2xyz}{2+xyz}-1$$

$$\Leftrightarrow\frac{\sum_{cyc}((a+1)(b+1))-\sum_{cyc}(a+1)}{\sum_{cyc}a+3}\ge\frac{(a+1)(b+1)(c+1)-1}{2+(a+1)(b+1)(c+1)}$$

$$(x=a+1,b=y+1,c=z+1)\Leftrightarrow\frac{\sum_{cyc}ab+\sum_{cyc}a}{\sum_{cyc}a+3}\ge\frac{\sum_{cyc}a+\sum_{cyc}ab+abc}{3+abc+\sum_{cyc}a+\sum_{cyc}ab}$$

$$\Leftrightarrow 3\sum_{cyc}ab+3\sum_{cyc}a+abc\sum_{cyc}ab+abc\sum_{cyc}a+2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+\left(\sum_{cyc}a\right)^2+$$

$$\left(\sum_{cyc}ab\right)^2\ge\left(\sum_{cyc}a\right)^2+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+abc\sum_{cyc}a+3\sum_{cyc}a+3\sum_{cyc}ab+$$

$$3abc\Leftrightarrow abc\sum_{cyc}ab+\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+\left(\sum_{cyc}ab\right)^2\overset{(*)}{\ge}3abc$$

$$\text{We shall now prove that : for m, n, p}\ge 0\text{, we have : }\sum_{cyc}m\ge 3.\sqrt[3]{mnp}\to(1)$$

$$\text{Now, if m, n, p}>0\text{, then, via AM}-\text{GM, }\sum_{cyc}m\ge 3.\sqrt[3]{mnp}\text{ and if :}$$

$$\text{exactly one or exactly two variables}=0\text{, then : }\sum_{cyc}m>0=3.\sqrt[3]{mnp}\text{ and if :}$$

$$m=n=p=0\text{, then : }\sum_{cyc}m=3.\sqrt[3]{mnp}=0\text{ and hence, (1) is true }\forall\text{ m, n, p}\ge 0$$

$$\text{and plugging in : m}=a\text{, n}=b\text{, p}=c\text{ and m}=ab\text{, n}=bc\text{, p}=ca\text{ successively,}$$

$$\text{we get : }\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\ge\left(3.\sqrt[3]{abc}\right)\left(3.\sqrt[3]{a^2b^2c^2}\right)=9abc$$

$$\therefore \text{LHS of } (*) - \text{RHS of } (*) = abc\sum_{\text{cyc}} ab + 6abc + \left(\sum_{\text{cyc}} ab\right)^2 \geq 0 \ (\because a,b,c \geq 0)$$

$$\therefore (*) \text{ is true} \Rightarrow 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+xy+yz+zx \geq 2(x+y+z)+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$$

$$\forall\, x,y,z \geq 1, '' ='' \text{ iff } x=y=z=1 \text{ (QED)}$$

Solution 2 by proposer

If $x,y \geq 1 \Rightarrow xy \geq 1 \Rightarrow xy-1 \geq 0; x-1 \geq 0; y-1 \geq 0$

We will prove that:

$$\frac{1}{x}+\frac{1}{y}+xy \geq x+y+\frac{1}{xy} \qquad (1)$$

$$y+x+x^2y^2 \geq x^2y+xy^2+1, \qquad x^2y^2-1+x-x^2y+y-xy^2 \geq 0$$

$$(xy-1)(xy+1)-x(xy-1)-y(xy-1) \geq 0$$

$$(xy-1)(xy+1-x-y) \geq 0$$

$$(xy-1)\big(x(y-1)-(y-1)\big) \geq 0$$

$$(xy-1)(y-1)(x-1) > 0 \text{ (true)}$$

Analogous with (1):

$$\frac{1}{y}+\frac{1}{z}+yz \geq y+z+\frac{1}{yz} \qquad (2)$$

$$\frac{1}{z}+\frac{1}{x}+xz \geq z+x+\frac{1}{xz} \qquad (3)$$

By adding (1); (2); (3):

$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+xy+yz+zx \geq 2(x+y+z)+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{xz}$$

Equality holds for $x=y=z$.

SP.584 Let be the sequence $(x_n)n \geq 1$ defined by

$$x_1 = 1, x_{n+2} = 3x_{n+1} - x_n, \forall n \in \mathbb{N}. \text{ Find:}$$

$$L = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{n} \frac{x_{2k+1}}{x_k + x_{k+1}}}$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by proposers

The sequence $(x_n)_{n\geq 0}$ verify a recurrence relationship by second order, then

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right], \forall n \in \mathbb{N}.$$

We show that: $x_1 + x_3 + x_5 + \cdots + x_{2n-1} = x_n^2, \forall n \in \mathbb{N}^*$. We have:

$$x_1 + x_3 + x_5 + \cdots + x_{2n-1} = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\left(\frac{3+\sqrt{5}}{2}\right)^{2k-1} - \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\left(\frac{3-\sqrt{5}}{2}\right)^{2k-1}$$

Using the formula of a geometrical progression, it follows:

$$x_1 + x_3 + x_5 + \cdots + x_{2n-1} = \frac{1}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{2n} + \left(\frac{3-\sqrt{5}}{2}\right)^{2n} - 5\right] = x_n^2$$

$$\begin{cases} x_1 + x_3 + x_5 + \cdots + x_{2k-1} = x_k^2 \\ x_1 + x_3 + x_5 + \cdots + x_{2k+1} = x_{k+1}^2 \end{cases} \overset{(-)}{\Rightarrow} x_{2k+1} = x_{k+1}^2 - x_k^2 \text{ or } \frac{x_{2k+1}}{x_k + x_{k+1}} = x_{k+1} - x_k$$

$$\sum_{k=0}^{n}\frac{x_{2k+1}}{x_k + x_{k+1}} = \sum_{k=0}^{n}(x_{k+1} - x_k) = x_{n+1}$$

Therefore:

$$\lim_{n\to\infty}\sqrt[n]{x_{n+1}} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3+\sqrt{5}}{2}\right)^{n+1}\right)} =$$

$$= \lim_{n\to\infty}\left(\frac{3+\sqrt{5}}{2}\right)\cdot\sqrt[n]{\frac{1}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{2} - \left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)(3-\sqrt{5})\right]} = \frac{3+\sqrt{5}}{2}$$

SP.585 Let $f: [n-1, n] \to [n, n+1]$ be a continuous function such that

$$\int_{n-1}^{n}\left(1 + xf'(x)\right)dx \leq nf(n) - (n-1)f(n-1),$$

then prove:

$$\int_{n-1}^{n}\frac{dx}{f(x)} \leq \frac{2}{n+1}, n \in \mathbb{N}^*$$

Proposed by by Marian Ursărescu and Florică Anastase – Romania

Solution by proposer

$$\int_{n-1}^{n}\left(1+xf'(x)\right)dx \le nf(n)-(n-1)f(n-1) \Rightarrow$$

$$nf(n)-(n-1)f(n-1)-\int_{n-1}^{n} xf'(x)\,dx \ge 1$$

But, using Integration by Parts method, we have:

$$\int_{n-1}^{n} xf'(x) = xf(x)|_{n-1}^{n} - \int_{n-1}^{n} f(x)\,dx$$

$$\int_{n-1}^{n} xf'(x)\,dx = nf(n)-(n-1)f(n-1)-\int_{n-1}^{n} f(x)\,dx$$

$$\int_{n-1}^{n} f(x)\,dx = nf(n)-(n-1)f(n-1)-\int_{n-1}^{n} xf'(x)\,dx \ge 1$$

For all $x \in [n-1, n]$ and $n \in \mathbb{N}^*$ we have $\frac{(f(x)-n)(f(x)-n-1)}{f(x)} \le 0$ or

$$f(x)-(2n+1)+\frac{n(n+1)}{f(x)} \le 0 \Leftrightarrow \frac{1}{f(x)} \le \frac{2n+1}{n(n+1)} - \frac{1}{n(n+1)}f(x) \Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx \le \frac{2n+1}{n(n+1)}\int_{n-1}^{n}dx - \frac{1}{n(n+1)}\int_{n-1}^{n} f(x)\,dx \Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx \le \frac{2n+1}{n(n+1)}\left(n(n-1)\right) - \frac{1}{n(n+1)} \Leftrightarrow$$

$$\int_{n-1}^{n}\frac{1}{f(x)}dx \le \frac{2n+1}{n(n+1)} - \frac{1}{n(n+1)} \Leftrightarrow \int_{n-1}^{n}\frac{1}{f(x)}dx \le \frac{2}{n+1}$$

UNDERGRADUATE PROBLEMS

UP.571 Prove that is the least value of the constant $k > 2$ such that:

$$\left(a^k+b^k+c^k\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$$

for any positive real numbers a, b, c with at most one of them less than 1 and

$$a^5+b^5+c^5=3$$

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

By choosing $b = 1, c \in (0,1]$ and $a = \sqrt[5]{2 - c^5}$, the constraints are satisfied, and the inequality is equivalent to $E(c) \geq 0$, where

$$E(c) = \left(a^k + c^k + 1\right)\left(\frac{1}{a} + \frac{1}{c} + 1\right) - 9.$$

Note that $c = 1$ implies $a = 1$. From $a^5 + c^5 = 2$, we get

$$a'(c) = \frac{-c^4}{a^4}, \quad a'(1) = -1,$$

$$a''(c) = \frac{4c^4a'}{a^5} - \frac{4c^3}{a^4}, \quad a''(1) = -8,$$

and

$$E'(c) = k\left(a^{k-1}a' + c^{k-1}\right)\left(\frac{1}{a} + \frac{1}{c} + 1\right) + \left(a^k + c^k + 1\right)\left(\frac{-a'}{a^2} - \frac{1}{c^2}\right),$$

$$E''(c) = k\left[(k-1)a^{k-2}(a')^2 + a^{k-1}a'' + (k-1)c^{k-2}\right]\left(\frac{1}{a} + \frac{1}{c} + 1\right)$$

$$+2k\left(a^{k-1}a' + c^{k-1}\right)\left(\frac{-a'}{a^2} - \frac{1}{c^2}\right) + \left(a^k + c^k + 1\right)\left(\frac{-a''}{a^2} + \frac{2(a')^2}{a^3} + \frac{2}{c^3}\right).$$

We have $E(1) = 0, E'(1) = 0$ and

$$E''(1) = 3k(k - 1 - 8 + k - 1) + 0 + 3(8 + 2 + 2) = 6(k-2)(k-3).$$

Since $E(1) = E'(1) = 0$, the condition $E''(1) \geq 0$ is necessary to have $E(c) \geq 0$ for $c \in (0,1]$. This condition implies $k \geq 3$. To show that 3 is the least value of the constant k. we need to show that $F \geq 0$, where

$$F = (a^3 + b^3 + c^3)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9.$$

Let

$$a \geq b \geq 1 \geq c.$$

For fixed a, assume that c and F are functions of b. By differentiating the equality constraint, we get

$$b^4 + c^4a' = 0, \quad c' = -b^4c^{-4},$$

therefore

$$F'(b) = 3(b^2 + c^2c')\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + (a^3 + b^3 + c^3)\left(\frac{-1}{b^2} + \frac{-c'}{c^2}\right)$$

$$= 3(b^2 - b^4c^{-2})\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + (a^3 + b^3 + c^3)\left(\frac{-1}{b^2} + b^4c^{-6}\right).$$

We will show that $F'(b) \geq 0$. Denoting $x = \frac{b}{c}$, we have

$$F'(b) = -3c^2(x^4 - x^2)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + b^{-2}(a^3 + b^3 + c^3)(x^6 - 1).$$

Since $x \geq 1$ and $a \geq b$, we have

$$F'(b) = -3c^2(x^4 - x^2)\left(\frac{2}{b} + \frac{1}{c}\right) + b^{-2}(2b^3 + c^3)(x^6 - 1),$$

hence

$$\frac{F'(b)}{c} \geq -3(x^4 - x^2)\left(\frac{2}{x} + 1\right) + \left(2x + \frac{1}{x^2}\right)(x^6 - 1)$$

$$= \frac{(x^2 - 1)(2x^7 + 2x^5 - 2x^4 - 4x^3 + x^2 + 1)}{x^2}$$

$$= \frac{(x^2 - 1)(x - 1)(2x^6 + 2x^5 + 4x^4 + 2x^3 - 2x^2 - x - 1)}{x^2} \geq 0.$$

From $F'(b) \geq 0$, it follows that $F(b)$ is increasing and has the minimum value when b is minimum, hence when $b = 1$. Thus, it suffices to prove the desired inequality for $b = 1$. We need to show that

$$(a^3 + c^3 + 1)\left(\frac{1}{a} + \frac{1}{c} + 1\right) \geq 9$$

for $a \geq 1 \geq c > 0$ such that $a^5 + c^5 = 2$. Let $S = \frac{a+c}{2} > \frac{1}{2}$ and $p = ac$. From the known inequality $a^5 + c^5 \geq 2S^5$, we get $S \leq 1$, and from

$$2 = a^5 + c^5 = (a^2 + c^2)(a^3 + c^3) - a^2c^2(a + c) = (4S^2 - 2p)(8S^3 - 6Sp) - 2Sp^2,$$

we obtain

$$5Sp^2 = 20S^3p - 16S^5 + 1,\ \ p = 2S^2 - \sqrt{\frac{4S^5 + 1}{5S}}.$$

We claim that

$$p \leq \frac{5S^2 - 1}{4},$$

that is

$$\sqrt{\frac{4S^5+1}{5S}} \ge \frac{3S^2+1}{4}.$$

By squaring, the inequality becomes

$$16(4S^5+1) \ge 5S(3S^2+1)^2, \quad 19S^5 - 30S^3 - 5s + 16 \ge 0,$$

$$(S-1)^2(19S^3+38S^2+27S+16) \ge 0$$

Write now the desired inequality as follows:

$$(8S^3 - 6Sp + 1)\left(\frac{2S}{p}+1\right) \ge 9, \;\; 3Sp^2 + 2(2+3S^2-2S^3)p - S(8S^3+1) \le 0,$$

$$\frac{3(20S^3p - 16S^5 + 1)}{5} + 2(2+3S^2-2S^3)p - S(8S^3+1) \le 0,$$

$$48S^5 + 40S^4 + 5S - 3 \ge 10(4S^3+3S^2+2)p.$$

Since $p \le \frac{5S^2-1}{4}$, we have

$$48S^5 + 40S^4 + 5S - 3 - 10(4S^3+3S^2+2)p \ge$$

$$\ge 48S^5 + 40S^4 + 5S - 3 - \frac{5(4S^3+3S^2+2)(5S^2-1)}{2}$$

$$= \frac{4 + 10S - 35S^2 + 20S^3 + 5S^4 - 4S^5}{2} = \frac{(1-S)^2(4+18S-3S^2-4S^3)}{2} \ge 0.$$

For $k = 3$ the equality occurs when $a = b = c = 1$.

UP.572 Prove that $25/17$ is the largest positive value of the constant k such that:

$$\frac{1}{a^2+k} + \frac{1}{b^2+k} + \frac{1}{c^2+k} + \frac{1}{d^2+k} \ge \frac{4}{1+k}$$

for any nonnegative real numbers a, b, c, d with at most one of them larger than 1 and $ab + ac + ad + bc + bd + cd = 6$.

Proposed by Vasile Cîrtoaje – Romania

Solution by proposer

Without loss of generality, assume that $a, b, c \in [0,1]$ and $d \geq 1$. For $b = c = 1$, the inequality becomes

$$\frac{1}{a^2+k} + \frac{1}{d^2+k} \geq \frac{2}{1+k},$$

under the constraint $ad + 4S = 5$, where $S = \frac{a+d}{2}$. From $5 = ad + 4S \leq S^2 + 4S$ and $5 = ad + 4S \geq 4S$, we get $1 \leq S \leq \frac{5}{4}$. Write the inequality as follows:

$$\frac{4S^2 - 2ad + 2k}{(ad-k)^2 + 4kS^2} \geq \frac{2}{1+k},$$

$$\frac{4S^2 - 2(5-4S) + 2k}{(5-k-4S)^2 + 4kS^2} \geq \frac{2}{1+k},$$

$$(S-1)[15 - 3k - (k+7)S] \geq 0$$

Choosing $S = \frac{5}{4}$, we get the necessary condition $k \leq \frac{25}{17}$. So, we only need to prove the original inequality for $k = \frac{25}{17}$.

If c, d are fixed, then the expression

$$F = \frac{1}{a^2+k} + \frac{1}{b^2+k} + \frac{1}{c^2+k} + \frac{1}{d^2+k}$$

has the minimum value when

$$E(a,b) = \frac{1}{a^2+k} + \frac{1}{b^2+k}$$

has the minimum value. Denoting

$$x = a, y = b, A = c + d, B = 6 - cd,$$

we have $A > 0, B \geq 0, x, y \in [0,1]$ and $xy + A(x+y) = B$. For $B = 0$, we have $x = y = 0$ (hence $a = b = 0$), while for $B > 0$, by Lemma below, it follows that $E(a,b)$ (hence F, too) has the minimum value for $\min\{a,b\} = 0$ or $a = b$. By extending this result to any two of a, b, c, it suffices to prove the original inequality for $a = b = c$, for $a = 0$ and $b = c$, and for $a = b = 0$.

***Case 1:* $a = b = c$. We need to show that**

$$\frac{3}{c^2+k}+\frac{1}{d^2+k}\ge\frac{4}{1+k}$$

for $0\le c\le 1\le d$ such that $c^2+cd=2$. The inequality is equivalent to

$$(1-c^2)^2(3-k-c^2)\ge 0.$$

It is true because $3-k-c^2\ge 2-k>0$.

Case 2: $a=0$ and $b=c$. We need to show that:

$$\frac{1}{k}+\frac{2}{c^2+k}+\frac{1}{d^2+k}\ge\frac{4}{1+k}$$

for $0\le c\le 1\le d$ such that $c^2+2cd=6$. Write the inequality as follows:

$$\frac{2}{c^2+k}-\frac{2}{1+k}\ge\frac{2}{1+k}-\frac{1}{k}-\frac{1}{d^2+k}$$

$$\frac{2(1-c^2)}{(1+k)(c^2+k)}\ge\frac{(k-1)(1-c^2)(36-c^2)}{k(1+k)[c^4-4(3-k)c^2+36]}$$

It is true if

$$\frac{2}{c^2+k}\ge\frac{(k-1)(36-c^2)}{k[c^4-4(3-k)c^2+36]},$$

i.e.

$$\frac{25}{c^2+k}\ge\frac{4(36-c^2)}{c^4-4(3-k)c^2+36}$$

It is true if

$$\frac{24}{1+k}\ge\frac{4\cdot 36}{0-4(3-k)+36},$$

i.e.

$$\frac{2}{1+k}\ge\frac{3}{6+k},\quad 9\ge k.$$

Case 3: $a=b=0$. We need to show that

$$\frac{2}{k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}\ge\frac{4}{1+k}$$

for $0<c\le 1\le d$ such that $cd=6$. It is true if

$$\frac{2}{k}+\frac{1}{1+k}+0\ge\frac{4}{1+k},$$

i.e. $k\le 2$.

The proof is completed. For $k=\frac{25}{17}$, the equality occurs when $a=b=c=d=1$, and also for $a=0,b=c=1$ and $d=\frac{5}{2}$ (or any cyclic permutation).

Lemma. Let A and B be positive real constants, and let $x,y\in[0,1]$ such that

$$xy+A(x+y)=B.$$

If $k>1$, then the expression

$$E=\frac{1}{x^2+k}+\frac{1}{y^2+k}$$

has the minimum value for $\min\{x,y\}=0$ or $x=y$.

Proof.

Let $s=x+y$ and $p=xy$. We need to show that if

$$0\le 4p\le s^2$$

and

$$p+As=B,$$

then the expression

$$E=\frac{s^2-2p+2k}{ks^2+(p-k)^2}$$

has the minimum value for $p=0$ (when $\min\{x,y\}=0$) or $4p=s^2$ (when $x=y$). From

$$B=p+As\ge p+2A\sqrt{p},$$

we get

$$p\le p_1=\left(\sqrt{A^2+B}-A\right)^2,$$

with equality for $4p=s^2$. Since

$$kE=1+\frac{k^2-p^2}{ks^2+(p-k)^2}=1+F(p),$$

where

$$F(p)=\frac{A^2(k^2-p^2)}{(A^2+k)p^2-2k(A^2+B)p+k^2A^2+kB^2},$$

it suffices to show that $F(p)$ has the minimum value m when $p=0$ or $p=p_1$. Write the inequality $F(p)\ge m$ as $F_1(p)\ge 0$, where

$$F_1(p) = -[(m+1)A^2 + km]p^2 + 2km(A^2 + B)p - (m-1)k^2A^2 - kmB^2.$$

Since $F_1(p)$ is concave on $[0, p_1]$, it has the minimum value 0 when $p = 0$ or $p = p_1$.

UP.573 Let be $\lambda > 1$ fixed. Find:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{k+1}{\lambda^k}$$

Proposed by Marin Chirciu – Romania

Solution by proposer

We have the sum of the geometric progression:

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}, x \neq 1.$$

We derive the equality above:

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \left(\frac{x^{n+1} - 1}{x - 1}\right)' = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Multiplying by $x, x \neq 0$, the above equality:

$$x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}.$$

Replacing $x = \frac{1}{\lambda}$ we obtain:

$$\sum_{k=0}^{n} \frac{k+1}{\lambda^k} = \frac{1}{\lambda^k} = \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{3}{\lambda^3} + \cdots + \frac{n}{\lambda^n} = \frac{\frac{n}{\lambda^{n+2}} - \frac{n+1}{\lambda^{n+1}} + \frac{1}{\lambda}}{\left(\frac{1}{\lambda} - 1\right)^2}.$$

Because $\frac{n}{\lambda^{n+2}} \to 0$ and $\frac{n+1}{\lambda^{n+1}} \to 0$ it follows that $\frac{\frac{n}{\lambda^{n+2}} - \frac{n+1}{\lambda^{n+1}} + \frac{1}{\lambda}}{\left(\frac{1}{\lambda} - 1\right)^2} \to \frac{\lambda}{(\lambda-1)^2}$

Then:

$$\sum_{k=0}^{n} \frac{1}{\lambda^k} = 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^n} = \frac{1 - \frac{1}{\lambda^{n+1}}}{1 - \frac{1}{\lambda}} \to \frac{\lambda}{\lambda - 1}$$

We obtain $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{k}{\lambda^k} = \frac{\lambda}{(\lambda-1)^2}$ and $\lim\limits_{n\to\infty} \sum_{k=0}^{n} \frac{1}{10^k} = \frac{\lambda}{\lambda-1}$ so

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k+1}{\lambda^k}=\left(\frac{\lambda}{\lambda-1}\right)^2$$

UP.574 If $0\le a\le b\le 1$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{\sqrt{1+xyz}}\ge(b-a)^2\int_a^b\frac{dx}{\sqrt{1+x^3}}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be $f:[0,1]\to\mathbb{R};f(x)=\frac{1}{\sqrt{1+e^{3x}}}$

$$f'(x)=\frac{-3e^{3x}}{2(1+e^{3x})\sqrt{1+e^{3x}}};$$

$$f''(x)=\frac{-3e^{3x}(2e^{3x}-1)}{4(1+e^{3x})^2\sqrt{1+e^{3x}}}<0;x\in[0,1]$$

$$x\in[0,1]\Rightarrow 0\le x\le 1\Rightarrow 0\le 3x\le 3\Rightarrow 1\le e^{3x}\le e^3$$

$$2\le 2e^{3x}\le 2e^3\Rightarrow 1\le 2e^{3x}-1\le 2e^3-1$$

$$2e^{3x}-1\ge 1\Rightarrow 2e^{3x}-1>0\Rightarrow -3e^{3x}(2e^{3x}-1)<0$$

$f''(x)<0\Rightarrow f$ concave. By Jensen's inequality:

$$f(a)+f(b)+f(c)\le 3f\left(\frac{a+b+c}{3}\right)$$

$$\frac{1}{\sqrt{1+e^{3a}}}+\frac{1}{\sqrt{1+e^{3b}}}+\frac{1}{\sqrt{1+e^{3c}}}\le\frac{3}{\sqrt{1+e^{a+b+c}}}$$

Denote: $x=e^a;y=e^b;z=e^c\Rightarrow$

$$\frac{1}{\sqrt{1+x^3}}+\frac{1}{\sqrt{1+y^3}}+\frac{1}{\sqrt{1+z^3}}\le\frac{3}{\sqrt{1+xyz}}$$

$$\int_a^b\int_a^b\int_a^b\left(\frac{1}{\sqrt{1+x^3}}+\frac{1}{\sqrt{1+y^3}}+\frac{1}{\sqrt{1+z^3}}\right)dx\,dy\,dz\le 3\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{\sqrt{1+xyz}}$$

$$3(b-a)^2\int_a^b\frac{dx}{\sqrt{1+x^3}}\le 3\int_a^b\int_a^b\int_a^b\frac{dx\,dy\,dz}{\sqrt{1+xyz}}$$

Equality holds for $a=b$.

UP.575 Calculate:

$$2\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dy\,dx$$

Proposed by Said Attaoui – Oran – Algeria

Solution 1 by proposer

Firstly, we can express

$$\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dx\,dy=\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx$$

Secondly, it is straightforward to confirm that

$$\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1-y}{x^2+y^2}+\frac{\arctan\left(\frac{y}{x}\right)}{x}\right)=\frac{y(y-1)}{(x^2+y^2)^2}$$

Therefore

$$2\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dy\,dx=\int_0^1\left[\frac{1-y}{x^2+y^2}+\frac{\arctan\left(\frac{y}{x}\right)}{x}\right]_1^\infty dx=$$

$$=\int_0^1\left(\frac{\frac{\pi}{2}}{x}-\frac{\arctan\left(\frac{1}{x}\right)}{x}\right)dx=\int_0^1\frac{\arctan(x)}{x}\,dx=\int_0^1\sum_{n=0}^\infty\frac{(-1)^n x^{2n}}{2n+1}\,dx=$$

$$=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^2}=G$$

Solution 2 by proposer

Firstly, we can express

$$\int_0^1\int_0^\infty \frac{y(y+1)}{(x^2+(y+1)^2)^2}\,dx\,dy=\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx$$

Secondly, substituting y by $\frac{1}{y}$, we obtain

$$\int_0^1\int_1^\infty \frac{y(y-1)}{(x^2+y^2)^2}\,dy\,dx=\int_0^1\int_0^1\frac{\frac{1}{y}\left(\frac{1}{y}-1\right)}{\left(x^2+\frac{1}{y^2}\right)^2}\left(\frac{1}{y^2}dy\right)dx=\int_0^1\int_0^1\frac{(1-y)}{(1+x^2y^2)^2}\,dy\,dx=$$

$$=\int_0^1\int_0^1(1-y)\left(\sum_{n=1}^{\infty}(-1)^{n-1}nx^{2(n-1)}y^{2(n-1)}\right)dy\,dx=$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\int_0^1 x^{2n-2}dx\right)^2-\sum_{n}^{\infty}(-1)^{n-1}n\left(\int_0^1 x^{2n-2}\,dx\right)\left(\int_0^1 y^{2n-1}\,dy\right)=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(2n-1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(2n-1)(2n)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(2n-1)^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{(2n-1)^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(2n-1)^2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}.$$

Therefore:

$$2\int_0^1\int_0^{\infty}\frac{y(y+1)}{(x^2+(y+1)^2)^2}dy\,dx=G.$$

UP.576 Find a closed form:

$$\int_0^{\infty}\frac{\arctan(x)}{\sqrt[5]{x^{10}+1}}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote: $A=\int_0^{\infty}\frac{\arctan(x)}{\sqrt[5]{x^{10}+1}}dx.$

We also consider the integral: $B=\int_0^{\infty}\frac{\text{arccot}(x)}{\sqrt[5]{x^{10}+1}}dx.$

We have:

$$A+B=\int_0^{\infty}\frac{\arctan(x)+\text{arccot}(x)}{\sqrt[5]{x^{10}+1}}dx=\frac{\pi}{2}\int_0^{\infty}\frac{1}{\sqrt[5]{x^{10}+1}}dx$$

We are going to calculate the integral:

$$C=\int_0^{\infty}\frac{1}{\sqrt[5]{x^{10}+1}}dx.$$

We use the following definition of Euler's Beta function:

$$B(p,q)=\int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy.$$

In the C integral we make the following variable change: $x^{10}=y$.

We obtain:

$$C=\frac{1}{10}\int_0^\infty \frac{y^{-\frac{9}{10}}}{(1+y)^{\frac{1}{5}}}dy.$$

For $p=\frac{1}{10}$ and $q=\frac{1}{10}$ we can write:

$$C=\frac{1}{10}B\left(\frac{1}{10},\frac{1}{10}\right).$$

We use the known relationship:

$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where $\Gamma(a)$ is the Euler's Gamma function.

We can write:

$$C=\frac{1}{10}\frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

So we can write:

$$A+B=\frac{\pi}{20}\frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

In the B integral we make the variable change $x=\frac{1}{t}$ and we immediately obtain

$$B=A.$$

So we have:

$$2A=\frac{\pi}{20}\frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

We obtain the value of the integral required in the problem statement:

$$A = \frac{\pi}{40}\frac{\Gamma^2\left(\frac{1}{10}\right)}{\Gamma\left(\frac{1}{5}\right)}.$$

Thus, the problem is solved.

UP.577 Find a closed form:

$$\int_0^1 \frac{x^2 \ln x}{x^3 + x\sqrt{x} + 1}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I = \int_0^1 \frac{x^2 \ln x}{x^3 + x\sqrt{x} + 1}dx$$

In this integral we make the variable change: $x = z^{\frac{2}{3}}$.

We obtain:

$$I = \frac{4}{9}\int_0^1 \frac{z \ln z}{z^2 + z + 1}dz.$$

We have, successively:

$$I = \frac{4}{9}\int_0^1 \frac{(1-z)z \ln z}{1 - z^3}dz$$

$$I = \frac{4}{9}\left(\int_0^1 \frac{z \ln z}{1 - z^3}dz - \int_0^1 \frac{z^2 \ln z}{1 - z^3}dz\right)$$

$$I = \frac{4}{9}\left(\int_0^1 \sum_{n=0}^{\infty} z^{3n+1} \ln z\, dz - \int_0^1 \sum_{n=0}^{\infty} z^{3n+2} \ln z\, dz\right)$$

$$I = \frac{4}{9}\sum_{n}^{\infty}\left(\int_0^1 z^{3n+1} \ln z\, dz - \int_0^1 z^{3n+2} \ln z\, dz\right).$$

We will use the following relationship:

$$\int_0^1 x^a \ln x\, dx = -\frac{1}{(a+1)^2}, \text{ where } a \in \mathbb{R}, a \geq 0.$$

We obtain:

$$I=\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{1}{(3n+3)^2}-\frac{1}{(3n+2)^2}\right].$$

Or:

$$I=\frac{4}{9}\sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{(n+1)^2}-\frac{\frac{1}{9}}{\left(n+\frac{2}{3}\right)^2}\right].$$

We now use the following relationship:

$$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\psi_1(x)$ is the trigamma function.

We have:

$$I=\frac{4}{81}\left[\psi_1(1)-\psi_1\left(\frac{2}{3}\right)\right].$$

The following special value of triagamma function is known:

$$\psi_1(1)=\frac{\pi^2}{6}$$

We obtained the value of the integral required in the problem statement:

$$I=\frac{4}{81}\left[\frac{\pi^2}{6}-\psi_1\left(\frac{2}{3}\right)\right].$$

UP.578 Prove the equality:

$$\int_0^{\infty}\frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx=\frac{1}{8}\pi^2\sqrt{5-\frac{7}{\sqrt{2}}}$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_0^{\infty}\frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx;$$

$$A=\int_0^1 \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx=\int_0^1 \frac{(1-x)x^4\sqrt{x}\ln x}{1-x^8}dx$$

$$B=\int_1^\infty \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)}dx=\int_1^\infty \frac{(1-x)x^4\sqrt{x}\ln x}{1-x^8}dx$$

We consider the integral A.

We have:

$$A=\int_0^1 \frac{x^{\frac{9}{2}}\ln x}{1-x^8}dx-\int_0^1 \frac{x^{\frac{11}{2}}\ln x}{1-x^8}dx$$

We will calculate:

$$A_1=\int_0^1 \frac{x^{\frac{9}{2}}\ln x}{1-x^8}dx$$

We make the variable change: $x^8=y; x=y^{\frac{1}{8}}$

We obtain:

$$A_1=\frac{1}{64}\int_0^1 \frac{y^{-\frac{5}{16}}\ln y}{1-y}dy$$

We will use the following known relationship:

$\int_0^1 \frac{z^a\ln z}{1-z}dz=-\psi_1(a+1), a\in R\setminus\{-1,-2,\dots\}$, **where $\psi_1(x)$ is the trigamma function**

We obtain:

$$A_1=-\frac{1}{64}\psi_1\left(\frac{11}{16}\right)$$

We will calculate:

$$A_2=\int_0^1 \frac{x^{\frac{11}{2}}\ln x}{1-x^8}dx$$

Proceeding similarly, we obtain:

$$A_2=-\frac{1}{64}\psi_1\left(\frac{13}{16}\right)$$

We obtained the value of the integral A:

$$A=A_1-A_2=\frac{1}{64}\left[-\psi_1\left(\frac{11}{16}\right)+\psi_1\left(\frac{13}{16}\right)\right]$$

We conside the integral B. We make the variable change: $x = \frac{1}{t}$.

We obtain:

$$B = \int_0^1 \frac{(t-1)t^{\frac{1}{2}} \ln t}{1-t^8} dt$$

By proceeding similarly to the integral A, we obtain:

$$B = \frac{1}{64}\left[\psi_1\left(\frac{3}{16}\right) - \psi_1\left(\frac{5}{16}\right)\right]$$

Result:

$$I = A + B$$

$$I = \frac{1}{64}\left[-\psi_1\left(\frac{11}{16}\right) + \psi_1\left(\frac{13}{16}\right) + \psi_1\left(\frac{3}{16}\right) - \psi_1\left(\frac{5}{16}\right)\right]$$

We use the reflection formula:

$$\psi_1 + \psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{3}{16}\right) + \psi_1\left(\frac{13}{16}\right) = \frac{\pi^2}{\sin^2 \frac{3\pi}{16}};\ \psi_1\left(\frac{5}{16}\right) + \psi_1\left(\frac{11}{16}\right) = \frac{\pi^2}{\sin^2 \frac{5\pi}{16}};$$

Result:

$$I = \frac{\pi^2}{64}\left(\frac{1}{\sin^2 \frac{3\pi}{16}} - \frac{1}{\sin^2 \frac{5\pi}{16}}\right)$$

We have:

$$\frac{1}{\sin^2 \frac{3\pi}{16}} - \frac{1}{\sin^2 \frac{5\pi}{16}} = 32 \sin^3 \frac{\pi}{8}$$

We propose to the reader to prove this equality.

Result:

$$I = \frac{1}{2}\pi^2 \sin^3 \frac{\pi}{8}$$

We have:

$$\sin\frac{\pi}{8}=\frac{1}{2}\sqrt{2-\sqrt{2}}$$

We propose to the reader to prove this equality.

Result:

$$I=\frac{1}{8}\pi^2\sqrt{5-\frac{7}{\sqrt{2}}}$$

Thus, the problem is solved.

UP.579 Find a closed form:

$$\int_{-1}^{1}\frac{1}{\sqrt[5]{(1-x)^2(1+x)^3}}dx$$

Proposed by Vasile Mircea Popa – Romania

Solution by proposer

Let us denote:

$$I=\int_{-1}^{1}\frac{1}{\sqrt[5]{(1-x)^2(1+x)^3}}dx$$

In this integral we make the variable change: $1+x=2t$.

We obtain:

$$I=\int_0^1 t^{-\frac{3}{5}}(1-t)^{-\frac{2}{5}}dt.$$

We use Euler's Beta function:

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\text{ where } p,q>0$$

We set conditions:

$$p-1=-\frac{3}{5};\, q-1=-\frac{2}{5}$$

Result:

$$p=\frac{2}{5};\, q=\frac{3}{5}$$

So:

$$I = B\left(\frac{2}{5}, \frac{3}{5}\right) = \frac{\Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)}{\Gamma(1)} = \Gamma\left(\frac{2}{5}\right)\Gamma\left(\frac{3}{50}\right)$$

where $\Gamma(a)$ is Euler's Gamma function.

We use the complement formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

We have:

$$I = \frac{\pi}{\sin\left(\frac{2\pi}{5}\right)}$$

The relationship is known:

$$\sin\left(\frac{2\pi}{5}\right) = \frac{1}{4}\sqrt{10+2\sqrt{5}}$$

We obtain:

$$I = \frac{4}{\sqrt{10+2\sqrt{5}}}\pi$$

After some elementary calculations we arrive at:

$$I = \sqrt{2-\frac{2}{\sqrt{5}}}\pi$$

We obtained the value of the integral required in the problem statement.

UP.580 If $f: [a,b] \to [-1,\infty)$; $a,b \in \mathbb{R}$; $a \le b$; f continuous, then:

$$\left(\int_a^b (1+f(x))dx\right)^3 \ge (b-a)^3 + 3(b-a)^2 \int_a^b f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

First we recall Bernoulli's generalized inequality:

If $n \in \mathbb{N}^*$; $x_1, x_2, \dots, x_n \in [-1,\infty)$; $x_i \cdot x_j \ge 0$; $(\forall) i,j \in \overline{1,n}$ then:

$$(1+x_1)(1+x_2)\cdot \ldots \cdot (1+x_n) \ge 1 + x_1 + x_2 + \cdots + x_n \quad (1)$$

For $n=2$ we must prove that:

$$(1+x_1)(1+x_2)\geq 1+x_1+x_2$$

$$1+x_1+x_2+x_1x_2\geq 1+x_1+x_2$$

$$x_1x_2\geq 0 \quad \text{(True)}$$

$P(n):\prod_{i=1}^{n}(1+x_i)\geq 1+\sum_{i=1}^{n}x_i$ **(suppose true)**

$P(n+1):\prod_{i=1}^{n+1}(1+x_i)\geq 1+\sum_{i=1}^{n+1}x_i$ **(to prove)**

$$\prod_{i=1}^{n+1}(1+x_i)=(1+x_{n+1})\cdot\prod_{i=1}^{n}(1+x_i)\overset{P(n)}{\geq}$$

$$\geq(1+x_{n+1})\cdot\left(1+\sum_{i=1}^{n}x_i\right)=$$

$$=1+\sum_{i=1}^{n}x_i+x_{n+1}+x_{n+1}\cdot\sum_{i=1}^{n}x_i\geq 1+\sum_{i=1}^{n+1}x_i$$

$$\sum_{i=1}^{n+1}x_i+\sum_{i=1}^{n}x_i\cdot x_{n+1}\geq\sum_{i=1}^{n+1}x_i$$

$$\sum_{i=1}^{n}x_i\cdot x_{n+1}\geq 0 \quad \text{(true)}$$

$$P(n)\to P(n+1)$$

We take in (1):

$$x_1=f(x);x_2=f(y);x_3=f(z);x_1,x_2,x_3\in[-1,\infty)$$

$$\big(1+f(x)\big)\big(1+f(y)\big)\big(1+f(z)\big)\geq 1+f(x)+f(y)+f(z)$$

By integrating:

$$\int_a^b\int_a^b\int_a^b\big(1+f(x)\big)\big(1+f(y)\big)\big(1+f(z)\big)dxdydz\geq$$

$$\geq\int_a^b\int_a^b\int_a^b\big(1+f(x)+f(y)+f(z)\big)\,dxdydz$$

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)\left(\int_a^b f(z)\,dz\right)\geq\int_a^b\int_a^b\int_a^b dx\,dy\,dz+$$

$$+\int_a^b\int_a^b\int_a^b f(x)\,dx\,dy\,dz+\int_a^b\int_a^b\int_a^b f(y)\,dx\,dy\,dz+\int_a^b\int_a^b\int_a^b f(z)\,dx\,dy\,dz$$

$$\left(\int_a^b f(x)\,dx\right)^3 \ge (b-a)^3 + 3\int_a^b f(x)\,dx \cdot \int_a^b \int_a^b dy\,dz$$

$$\left(\int_a^b f(x)\,dx\right)^3 \ge (b-a)^3 + 3(b-a)^2 \int_a^b f(x)\,dx$$

Equality holds for $a=b$.

UP.581 If $1 \le a \le b$ then:

$$24(b-a)^2 \ln\frac{b}{a} + (b^2-a^2)^3 \ge 12(b-a)^2(b^2-a^2) + 24\ln^3\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

The inequality can be written:

$$3(b-a)^2 \ln\frac{b}{a} + \left(\frac{b^2-a^2}{2}\right)^3 \ge 3(b-a)^2\left(\frac{b^2-a^2}{2}\right) + \ln^3\left(\frac{b}{a}\right)$$

$$3\int_a^b dx \cdot \int_a^b dy \cdot \int_a^b \frac{1}{z}\,dz + \left(\int_a^b x\,dx\right)^3 \ge$$

$$\ge 3\int_a^b dx \int_a^b dy \int_a^b z\,dz + \int_a^b \frac{1}{x}\,dx \int_a^b \frac{1}{y}\,dy + \int_a^b \frac{1}{z}\,dz$$

$$3\int_a^b \int_a^b \int_a^b \frac{1}{x}\,dx\,dy\,dz + \int_a^b \int_a^b \int_a^b xyz\,dx\,dy\,dz \ge$$

$$\ge 3\int_a^b \int_a^b \int_a^b x\,dx\,dy\,dz + \int_a^b \int_a^b \int_a^b \frac{1}{xyz}\,dx\,dy\,dz$$

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)dx\,dy\,dz \ge$$

$$\ge \int_a^b \int_a^b \int_a^b \left(x+y+z+\frac{1}{xyz}\right)dx\,dy\,dz$$

It is enough to prove that:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz \ge x+y+z+\frac{1}{xyz} \qquad (1)$$

We will prove by mathematical induction that:

$$P(n):\sum_{i=1}^{n}\frac{1}{x_i}+x_1x_2\dots x_n\ge\sum_{i=1}^{n}x_i+\frac{1}{x_1x_2\cdot\ldots\cdot x_n}$$

For $x_1=x;x_2=y;x_3=z;n=3$ we obtain (1) by (2).

Suppose $P(n)$ true:

$$P(n+1):\sum_{i=1}^{n+1}\frac{1}{x_i}+x_1x_2\dots x_nx_{n+1}\ge\sum_{i=1}^{n+1}x_i+\frac{1}{x_1x_2\cdot\ldots\cdot x_nx_{n+1}}\quad\text{(to prove)}$$

$$\sum_{i=1}^{n+1}\frac{1}{x_i}+x_1x_2\dots x_nx_{n+1}=\sum_{i=1}^{n}\frac{1}{x_i}+\frac{1}{x_{n+1}}+x_1x_2\dots x_nx_{n+1}=$$

$$=\sum_{i=1}^{n}\frac{1}{x_i}+x_1x_2\dots x_n-x_1x_2\cdot\ldots\cdot x_n+\frac{1}{x_{n+1}}+x_1x_2\dots x_nx_{n+1}\ge$$

$$\overset{P(n)}{\ge}\sum_{i=1}^{n}x_i+\frac{1}{x_1x_2\dots x_n}-x_1x_2\dots x_n+\frac{1}{x_{n+1}}+x_1x_2\dots x_nx_{n+1}$$

Denote $y=x_1x_2\cdot\ldots\cdot x_n$. Remains to prove that:

$$\sum_{i=1}^{n}x_i+\frac{1}{y}-y+\frac{1}{x_{n+1}}+yx_{n+1}\ge\sum_{i=1}^{n+1}x_i+\frac{1}{yx_{n+1}}$$

$$\frac{1}{y}-y+\frac{1}{x_{n+1}}+yx_{n+1}\ge x_{n+1}+\frac{1}{yx_{n+1}}$$

Let's observe that:

$$y\ge1;x_{n+1}\ge1;yx_{n+1}\ge1\text{ or}$$

$$y-1\ge0;x_{n+1}-1\ge0;yx_{n+1}-1\ge0\qquad(3)$$

$$yx_{n+1}-x_{n+1}+\frac{1}{x_{n+1}}-\frac{1}{yx_{n+1}}+\frac{1}{y}-y\ge0\quad\text{(to prove)}$$

$$x_{n+1}(y-1)+\frac{1}{x_{n+1}}\left(1-\frac{1}{y}\right)-\frac{y^2-1}{y}\ge0$$

$$x_{n+1}(y-1)+\frac{1}{x_{n+1}}\cdot\frac{y-1}{y}-\frac{(y-1)(y+1)}{y}\ge0$$

$$(y-1)\left(x_{n+1}+\frac{1}{yx_{n+1}}-\frac{y+1}{y}\right)\ge0$$

$$(y-1)\left(x_{n+1}-1+\frac{1}{yx_{n+1}}-\frac{1}{y}\right)\ge0$$

$$(y-1)\left(x_{n+1}-1-\frac{x_{n+1}-1}{yx_{n+1}}\right)\ge 0$$

$$(y-1)(x_{n+1}-1)\left(1-\frac{1}{yx_{n+1}}\right)\ge 0$$

$$(y-1)(x_{n+1}-1)(yx_{n+1}-1)\ge 0 \quad \text{(true by (3))}$$

Equality holds for $a=b$.

UP.582 Prove that exists $X\in M_{2,3}(\mathbb{R})$; $Y\in M_{3,2}(\mathbb{R})$ such that:

$$X\cdot Y=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}; Y\cdot X=\begin{pmatrix}2 & 6 & 6\\ 3 & 9 & 9\\ -3 & -9 & -9\end{pmatrix}$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

Let be:

$$X=\begin{pmatrix}1 & a & b\\ 1 & a & b\end{pmatrix}; Y=\begin{pmatrix}1 & 1\\ -b & 2b\\ a & -2a\end{pmatrix}; a,b\in\mathbb{R}$$

$$X\cdot Y=\begin{pmatrix}1 & a & b\\ 1 & a & b\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ -b & 2b\\ a & -2a\end{pmatrix}=$$

$$=\begin{pmatrix}1-ab+ab & 1+2ab-2ab\\ 1-ab+ab & 1+2ab-2ab\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$Y\cdot X=\begin{pmatrix}1 & 1\\ -b & 2b\\ a & -2a\end{pmatrix}=\begin{pmatrix}1 & a & b\\ 1 & a & b\end{pmatrix}=$$

$$=\begin{pmatrix}1\cdot 1+1\cdot 1 & 1\cdot a+1\cdot a & 1\cdot b+1\cdot b\\ -b\cdot 1+2b\cdot 1 & -b\cdot a+2b\cdot a & -b\cdot b+2b\cdot b\\ a\cdot 1-2a\cdot 1 & a\cdot a-2a\cdot a & a\cdot b-2a\cdot b\end{pmatrix}=$$

$$=\begin{pmatrix}2 & 2a & 2b\\ b & ab & b^2\\ -a & -a^2 & -ab\end{pmatrix}=\begin{pmatrix}2 & 6 & 6\\ 3 & 9 & 9\\ -3 & -9 & -9\end{pmatrix}$$

We take $a=3$; $b=3$ hence:

$$X=\begin{pmatrix}1 & 3 & 3\\ 1 & 3 & 3\end{pmatrix}; Y=\begin{pmatrix}1 & 1\\ -3 & 6\\ 3 & -6\end{pmatrix}$$

UP.583 Solve for complex numbers:

$$x^8+4x^7+22x^6+52x^5+69x^4+56x^3+28x^2+8x+1=0$$

Proposed by Daniel Sitaru – Romania

Solution by proposer

$$x^8+4x^7+22x^6+52x^5+69x^4+56x^3+28x^2+8x+1=0$$

$$x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1-x^8-4x^7-6x^6-4x^5-x^4+x^8=0$$

$$(x+1)^8-x^4(x^4+4x^3+6x^2+4x+1)+x^8=0$$

$$(x+1)^8-x^4(x+1)^4+x^8=0$$

$$\left(\frac{x+1}{x}\right)^8-\left(\frac{x+1}{x}\right)^4+1=0$$

$$\left(1+\frac{1}{x}\right)^8-\left(1+\frac{1}{x}\right)^4+1=0$$

Denote $\frac{1}{x}=y$

$$(1+y)^8-(1+y)^4+1=0$$

Denote $(1+y)^4=u$

$$u^2-u+1=0$$

$$u_1=\frac{1+i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$u_2=\frac{1-i\sqrt{3}}{2}=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$$

$$(1+y)^4=u_1$$

$$(1+y)^4=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$1+y=\sqrt[4]{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\{z_0,z_1,z_2,z_3\}$$

$$z_k=\cos\frac{\frac{\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{\pi}{3}+2k\pi}{4};k\in\overline{0,3}$$

$$z_k=\cos\frac{\pi+6k\pi}{12}+i\sin\frac{\pi+6k\pi}{12};k\in\overline{0,3}$$

$$(1+y)^4=u_2$$

$$1+y=\sqrt[4]{\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}}=\{z'_0,z'_1,z'_2,z'_3\}$$

$$z'_k=\cos\frac{\frac{5\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{5\pi}{3}+2k\pi}{4};k\in\overline{0,3}$$

$$z'_k=\cos\frac{5\pi+6k\pi}{12}+i\sin\frac{5\pi+6k\pi}{12};k\in\overline{0,3}$$

$$y_k=-1+z_k;y'_k=-1+z'_k;k\in\overline{0,3}$$

$$\frac{1}{x_k}=-1+z_k;\frac{1}{x'_k}=-1+z'_k;k\in\overline{0,3}$$

$$x_k=\frac{1}{-1+z_k};x'_k=\frac{1}{-1+z'_k};k\in\overline{0,3}$$

The solutions are:

$$x_k=\frac{1}{-1+\cos\frac{\pi+6k\pi}{12}+i\sin\frac{\pi+6k\pi}{12}};k\in\overline{0,3}$$

$$x'_k=\frac{1}{-1+\cos\frac{5\pi+6k\pi}{12}+i\sin\frac{5\pi+6k\pi}{12}};k\in\overline{0,3}$$

UP.584 **Let $a,b,c,d>1$ and $f:[a,b]\to[c,d]$ a continuous function for which $\exists\lambda\in(a,b)$ such that**

$$a\int_a^\lambda f(x)\,dx+b\int_\lambda^b f(x)\,dx\geq a+c$$

then prove:

$$\int_a^b\frac{x}{f(x)}dx\leq\left(\frac{1}{a}+\frac{1}{b}\right)\frac{b^2-a^2-2}{2}$$

Proposed by Marian Ursărescu and Florică Anastase – Romania

Solution by the proposers

Let $F:[a,b]\to\mathbb{R},F(t)=\int_a^t f(x)\,dx$. Because f is a continuous function, the function F is derivable and $F'(t)=f(t),\forall t\in[a,b]$ we have:

$$\int_a^b xf(x)\,dx \overset{IBP}{=} xF(x)|_a^b - \int_a^b F(x)\,dx = b\int_a^b f(x)\,dx - \int_a^b F(x)\,dx$$

Using Mean Value Theorem:

$$\exists\lambda \in (a,b) \text{ such that } \int_a^b F(x)\,dx = (b-a)F(\lambda) \text{ or}$$

$$\int_a^b xf(x)\,dx = b\int_a^b f(x)\,dx - (b-a)F(\lambda) =$$

$$= b\left(\int_a^\lambda f(x)\,dx + \int_\lambda^b f(x)\,dx\right) - (b-a)\int_a^\lambda f(x)\,dx = a\int_a^\lambda f(x)\,dx + b\int_\lambda^b f(x)\,dx$$

Therefore,

$$a\int_a^\lambda f(x)\,dx + b\int_\lambda^b f(x)\,dx = \int_a^b xf(x)dx \geq a+c \qquad \textbf{(1)}$$

On the other hands, we have:

$$\frac{\big(c-f(x)\big)\big(ax-xf(x)\big)}{f(x)} \leq 0, \forall x \in [a,b] \Leftrightarrow$$

$$\frac{acx}{f(x)} - (cx+ax) + xf(x) \leq 0 \Leftrightarrow \frac{x}{f(x)} \leq \left(\frac{1}{a}+\frac{1}{b}\right)x - \frac{1}{ac}xf(x) \Leftrightarrow$$

$$\int_a^b \frac{x}{f(x)}dx \leq \left(\frac{1}{a}+\frac{1}{c}\right)\int_a^b x\,dx - \frac{1}{ac}\int_a^b xf(x)\,dx \overset{(1)}{\Leftrightarrow}$$

$$\int_a^b \frac{x}{f(x)}dx \leq \left(\frac{1}{a}+\frac{1}{c}\right)\frac{b^2-a^2}{2} - \left(\frac{1}{a}+\frac{1}{c}\right) \Leftrightarrow \int_a^b \frac{x}{f(x)}dx \leq \left(\frac{1}{a}+\frac{1}{c}\right)\frac{b^2-a^2-2}{2}$$

UP.585 If $f:(0,\infty) \to (0,\infty)$; f continuous and

$$f\left(\frac{1}{x}\right) + f\left(\frac{1}{y}\right) = 2f\left(\frac{1}{x+y}\right); (\forall) x,y > 0 \text{ then } (\forall)\ a,b > 0:$$

$$\int_a^b\int_a^b\int_a^b f\left(\frac{1}{x+y+z}\right)dx\,dy\,dz = (b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$We\ know\ that \int_m^n f(p)dp = \int_m^n f(q)dq \ (1)$$

$$f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)=2f\left(\frac{1}{x+y}\right)\ \ or,\qquad f\left(\frac{1}{x+y}\right)=\frac{1}{2}\left(f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)\right)$$

Using this we get:

$$f\left(\frac{1}{x+y+z}\right)=f\left(\frac{1}{x+(y+z)}\right)=\frac{1}{2}\left(f\left(\frac{1}{x}\right)+f\left(\frac{1}{y+z}\right)\right)=$$

$$=\frac{1}{2}\left(f\left(\frac{1}{x}\right)+\frac{1}{2}\left(f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)\right)\right)=\frac{1}{2}f\left(\frac{1}{x}\right)+\frac{1}{4}f\left(\frac{1}{y}\right)+\frac{1}{4}f\left(\frac{1}{z}\right)\ (2)$$

$$\int_a^b\int_a^b\int_a^b f\left(\frac{1}{x+y+z}\right)dxdydz\overset{(2)}{=}\int_a^b\int_a^b\int_a^b\left(\frac{1}{2}f\left(\frac{1}{x}\right)+\frac{1}{4}f\left(\frac{1}{y}\right)+\frac{1}{4}f\left(\frac{1}{z}\right)\right)dxdydz=$$

$$=\frac{1}{2}\int_a^b\int_a^b\int_a^b f\left(\frac{1}{x}\right)dxdydz+\frac{1}{4}\int_a^b\int_a^b\int_a^b f\left(\frac{1}{y}\right)dxdydz+\frac{1}{4}\int_a^b\int_a^b\int_a^b f\left(\frac{1}{z}\right)dxdydz$$

$$=\frac{1}{2}(y)_a^b(z)_a^b\int_a^b f\left(\frac{1}{x}\right)dx+\frac{1}{4}(x)_a^b(z)_a^b\int_a^b f\left(\frac{1}{y}\right)dy+\frac{1}{4}(y)_a^b(x)_a^b\int_a^b f\left(\frac{1}{z}\right)dz\overset{(1)}{=}$$

$$=\frac{1}{2}(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx+\frac{1}{4}(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx+\frac{1}{4}(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx=$$

$$=(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx$$

Solution 2 by proposer

$$f\left(\tfrac{1}{x}\right)+f\left(\tfrac{1}{y}\right)=2f\left(\tfrac{1}{x+y}\right)\qquad(1)$$

$$f\left(\tfrac{1}{y}\right)+f\left(\tfrac{1}{z}\right)=2f\left(\tfrac{1}{y+z}\right)\qquad(2)$$

$$f\left(\tfrac{1}{z}\right)+f\left(\tfrac{1}{x}\right)=2f\left(\tfrac{1}{z+x}\right)\qquad(3)$$

By adding (1); (2); (3):

$$2\left(f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)\right)=2\left(f\left(\frac{1}{x+y}\right)+f\left(\frac{1}{y+z}\right)+f\left(\frac{1}{z+x}\right)\right)$$

$$f\left(\tfrac{1}{x}\right)+f\left(\tfrac{1}{y}\right)+f\left(\tfrac{1}{z}\right)=f\left(\tfrac{1}{x+y}\right)+f\left(\tfrac{1}{y+z}\right)+f\left(\tfrac{1}{z+x}\right)\qquad(4)$$

Replacing $y \to y+z$ in (1); $z \to z+x$ in (2); $x \to x+y$ in (3) we obtain:

$$f\left(\frac{1}{x}\right)+f\left(\frac{1}{y+z}\right)=2f\left(\frac{1}{x+y+z}\right) \qquad (5)$$

$$f\left(\frac{1}{y}\right)+f\left(\frac{1}{z+x}\right)=2f\left(\frac{1}{x+y+z}\right) \qquad (6)$$

$$f\left(\frac{1}{z}\right)+f\left(\frac{1}{x+y}\right)=2f\left(\frac{1}{x+y+z}\right) \qquad (7)$$

By adding (5); (6); (7):

$$f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)+f\left(\frac{1}{y+z}\right)+f\left(\frac{1}{z+x}\right)+f\left(\frac{1}{x+y}\right)=6f\left(\frac{1}{x+y+z}\right)$$

Using (4) we obtain:

$$f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)+f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)=6f\left(\frac{1}{x+y+z}\right)$$

$$2\left(f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)\right)=6f\left(\frac{1}{x+y+z}\right)$$

$$3f\left(\frac{1}{x+y+z}\right)=f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)$$

By integrating:

$$3\int_a^b\int_a^b\int_a^b f\left(\frac{1}{x+y+z}\right)dx\,dy\,dz=$$

$$=\int_a^b\int_a^b\int_a^b\left(f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+f\left(\frac{1}{z}\right)\right)dx\,dy\,dz=$$

$$=\int_a^b f\left(\frac{1}{x}\right)dx\int_a^b dy\int_a^b dz+\int_a^b dx+\int_a^b f\left(\frac{1}{y}\right)dy\int_a^b dz+\int_a^b dx\int_a^b dy\int_a^b f\left(\frac{1}{z}\right)dz=$$

$$=(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx+(b-a)^2\int_a^b f\left(\frac{1}{y}\right)dy+(b-a)^2\int_a^b f\left(\frac{1}{z}\right)dz=$$

$$=3(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx$$

$$\int_a^b\int_a^b\int_a^b f\left(\frac{1}{x+y+z}\right)dx\,dy\,dz=(b-a)^2\int_a^b f\left(\frac{1}{x}\right)dx$$