RMM - Cyclic Inequalities Marathon 1701 - 1800

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Proposed by

Daniel Sitaru-Romania, Neculai Stanciu-Romania

D.M.Bătinețu-Giurgiu-Romania, Claudia Nănuți-Romania

Marin Chirciu-Romania, Nguyen Hung Cuong-Vietnam

Zaza Mzhavanadze-Georgia, Mihaly Bencze-Romania

Gheorghe Crăciun-Romania, Pavlos Trifon-Greece

Phan Ngoc Chau-Vietnam

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Solutions by

Daniel Sitaru-Romania, Mohamed Amine Ben Ajiba-Morocco

Soumava Chakraborty-India, Tapas Das-India

Christos Tsifakis-Greece

1701. If $x,y,t>0$ then:

$$\frac{x^2+ty}{t+y}+\frac{y^2+tx}{t+x}\ge x+y$$

Proposed by D.M.Bătinețu-Giurgiu, Claudia Nănuți-Romania

Solution by Tapas Das-India

$$We\ need\ to\ show:$$

$$\frac{x^2+ty}{t+y}+\frac{y^2+tx}{t+x}\ge x+y$$

$$(x^2+ty)(t+x)+(y^2+tx)(t+y)\ge(x+y)(t+y)(t+x)$$

$$t(x^2+y^2)+t^2(x+y)+2txy+x^3+y^3\ge t^2(x+y)+t(x+y)^2+xy(x+y)$$

$$t(x^2+y^2+2xy)+x^3+y^3\ge t(x+y)^2+xy(x+y)$$

$$t(x+y)^2+x^3+y^3\ge\ t(x+y)^2+xy(x+y)$$

$$x^3+y^3\ge xy(x+y)\ true$$

$$as\ x^3+y^3=(x+y)(x^2+y^2-xy)\overset{AM-GM}{\ge}xy(x+y)$$

Equality holds for $x=y$.

1702. If $x,y,z>0, xyz\ge 1$ and $\lambda\ge 0$, then :

$$\sum_{cyc}\frac{x^{16}-x^4+3}{x^2y^2+\lambda z(x+y)}\ge\frac{9}{2\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}x^2y^2\ge\frac{1}{3}\left(\sum_{cyc}xy\right)\left(\sum_{cyc}xy\right)\overset{A-G}{\ge}\frac{1}{3}\left(3.\sqrt[3]{x^2y^2z^2}\right)\left(\sum_{cyc}xy\right)\overset{xyz\ge 1}{\ge}\sum_{cyc}xy$$

$$\overset{\because\lambda\ge 0}{\Rightarrow}\lambda\sum_{cyc}xy\le\lambda\sum_{cyc}x^2y^2\to(1)$$

$$\text{Again},x^{16}-x^4+3=(x^{16}+1+1+1)-x^4\overset{A-G}{\ge}4x^4-x^4$$

$$\Rightarrow x^{16} - x^4 + 3 \ge 3x^4 \text{ and analogs} \to (2)$$

$$\text{We have}: \sum_{cyc} \frac{x^{16} - x^4 + 3}{x^2y^2 + \lambda z(x+y)} \overset{\text{via (2)}}{\ge} 3\sum_{cyc} \frac{x^4}{x^2y^2 + \lambda z(x+y)} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{3\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^2y^2 + 2\lambda \sum_{cyc} xy} \overset{\text{via (1)}}{\ge} \frac{3\left(\sum_{cyc} x^2\right)^2}{\sum_{cyc} x^2y^2 + 2\lambda \sum_{cyc} x^2y^2} \ge$$

$$\ge \frac{9\sum_{cyc} x^2y^2}{(2\lambda+1)\sum_{cyc} x^2y^2} = \frac{9}{2\lambda+1}$$

$$\therefore \sum_{cyc} \frac{x^{16} - x^4 + 3}{x^2y^2 + \lambda z(x+y)} \ge \frac{9}{2\lambda+1}$$

$$\forall\, x,y,z > 0 \,\big|\, xyz \ge 1 \text{ and } \lambda \ge 0, \text{''}=\text{''} \text{ iff } x=y=z=1 \text{ (QED)}$$

1703. If $a,b,c > 0$, then prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a+b+c}{\sqrt{a^2+b^2+c^2}} \ge 3 + \sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2} \therefore \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a+b+c}{\sqrt{a^2+b^2+c^2}}$$

$$\ge \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2} + \frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}} \overset{?}{\ge} 3 + \sqrt{3} \Leftrightarrow \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2} - 3 \overset{?}{\ge} \sqrt{3} - \frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}$$

$$\Leftrightarrow \frac{9\sum_{cyc} a^2 - 3\sum_{cyc} a^2 - 6\sum_{cyc} ab}{\left(\sum_{cyc} a\right)^2} \overset{?}{\ge} \frac{3 - \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2}}{\sqrt{3} + \frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}}$$

$$\Leftrightarrow \frac{6\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}{\left(\sum_{cyc} a\right)^2} \boxed{\overset{?}{\underset{(*)}{\ge}}} \frac{2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}{\left(\sum_{cyc} a^2\right)\left(\sqrt{3} + \frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}\right)}$$

$$\text{Now, } \sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}} > 1 \text{ and}$$

$$\because \sum_{cyc} a^2 - \sum_{cyc} ab \ge 0$$

$$\therefore \frac{2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}{\left(\sum_{cyc} a^2\right)\left(\sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}\right)} \le \frac{2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}{\sum_{cyc} a^2} \le \frac{6\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)}{\left(\sum_{cyc} a\right)^2}$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{\sqrt{a^2+b^2+c^2}} \ge 3+\sqrt{3}\ \forall a,b,c>0,$$

$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$

1704. If $a,b,c \ge 1$ and $a+b+c=6$, then prove that :

$$(a^2+2)(b^2+2)(c^2+2) \le 216$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables equal to 1 and WLOG we may assume $b=c=1$ $(a=4)$ and then : $(a^2+2)(b^2+2)(c^2+2)=162<216$

$\boxed{\text{Case 2}}$ Exactly 1 variable equals to 1 and WLOG we may assume $a=1$ $\left(b,c>1 \,\middle|\, b+c=5\right)$ and then : $(a^2+2)(b^2+2)(c^2+2)$

$$=3(b^2+2)((5-b)^2+2) \overset{?}{<} 216 \Leftrightarrow b^4-10b^3+29b^2-20b-18 \overset{?}{<} 0$$

$$\Leftrightarrow \frac{b(b-5)(2b-5)^2}{4}-\frac{9(2b-5)^2}{16}-\frac{63}{16} \overset{?}{<} 0 \to \text{true} \because b=5-c \overset{c>1}{<} 4<5$$

$$\therefore (a^2+2)(b^2+2)(c^2+2)<216$$

$\boxed{\text{Case 3}}$ $a,b,c>1$ and then : $a=x+1, b=y+1, c=z+1$ where $x,y,z>0$ such that : $\sum_{cyc} x = 3$ and assigning $y+z=M, z+x=N, x+y=P$

$$\Rightarrow M+N-P=2z>0, N+P-M=2x>0 \text{ and } P+M-N=2y>0$$

$\Rightarrow M+N>P, N+P>M, P+M>N \Rightarrow M,N,P$ form sides of a triangle with semiperimeter, circumradius and inradius $= s,R,r$ (say) yielding

$$2\sum_{cyc} x = \sum_{cyc} M = 2s \Rightarrow \sum_{cyc} x = s \to (1) \Rightarrow x=s-M, y=s-N, z=s-P$$

$$\Rightarrow xyz = r^2 s \rightarrow (2) \text{ and via such substitutions, } \sum_{cyc} xy = \sum_{cyc} (s - M)(s - N)$$

$$\Rightarrow \sum_{cyc} xy = 4Rr + r^2 \rightarrow (3) \text{ and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy$$

$$= \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\therefore \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \rightarrow (4) \text{ and finally, } \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s^2 \Rightarrow \sum_{cyc} x^2y^2 = r^2((4R + r)^2 - 2s^2) \rightarrow (5)$$

$$\text{Now, } (a^2 + 2)(b^2 + 2)(c^2 + 2) - 216 = \prod_{cyc} (x^2 + 2x + 3) - 216$$

$$= x^2y^2z^2 + 2xyz\sum_{cyc} xy + 3\sum_{cyc} x^2y^2 + 4xyz\sum_{cyc} x + 6\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 18xyz$$

$$+8xyz + 9\sum_{cyc} x^2 + 12\sum_{cyc} xy + 18\sum_{cyc} x + 27 - 216 \overset{\sum_{cyc} x = 3}{=}$$

$$x^2y^2z^2 + \frac{2}{3}xyz\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + \frac{3}{9}\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2y^2\right) + \frac{4}{9}xyz\left(\sum_{cyc} x\right)^3$$

$$+\frac{6}{27}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} xy\right) - \frac{10}{27}xyz\left(\sum_{cyc} x\right)^3 + \frac{9}{81}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} x^2\right)$$

$$+\frac{12}{81}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} xy\right) + \frac{18}{243}\left(\sum_{cyc} x\right)^6 - \frac{189}{729}\left(\sum_{cyc} x\right)^6 \overset{?}{\leq} 0$$

$$\overset{\text{via (1),(2),(3),(4) and (5)}}{\Leftrightarrow} 5s^4 - 10(4Rr + r^2)s^2 - 3s^2(s^2 - 8Rr - 2r^2) - 2r^2s^2$$

$$-9r^2((4R + r)^2 - 2s^2) - 18r^2(4Rr + r^2) - 27r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow s^4 - (8Rr - 6r^2)s^2 - r^2(72R^2 + 72Rr + 27r^2) \underset{(*)}{\overset{?}{\geq}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (8Rr + r^2)s^2 - r^2(72R^2 + 72Rr + 27r^2) \overset{\text{Gerretsen}}{\geq}$$

$$(8Rr + r^2)(16Rr - 5r^2) - r^2(72R^2 + 72Rr + 27r^2) = 8(R - 2r)(7R + 2r) \overset{\text{Euler}}{\geq} 0$$

$$\Rightarrow (*) \text{ is true } \therefore (a^2 + 2)(b^2 + 2)(c^2 + 2) \leq 216 \text{ and so, combining all 3 cases,}$$

$$(a^2+2)(b^2+2)(c^2+2)\le 216\ \forall\, a,b,c\ge 1 \,\middle|\, a+b+c=6,$$
$$''='' \text{ iff } a=b=c=2 \text{ (QED)}$$

1705. **If $a,b,c>0,\lambda\ge 0$ then:**

$$\sum\frac{(a+\lambda b)(a+\lambda c)}{bc}\ge 3(\lambda+1)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{(a+\lambda b)(a+\lambda c)}{bc}=\sum\left(\left(\frac{a}{b}+\lambda\right)\left(\frac{a}{c}+\lambda\right)\right)\overset{C-S}{\ge}\sum\left(\frac{a}{\sqrt{bc}}+\lambda\right)^2\overset{CBS}{\ge}$$

$$\ge\frac{1}{3}\left(\frac{a}{\sqrt{bc}}+\frac{b}{\sqrt{ca}}+\frac{c}{\sqrt{ab}}+3\lambda\right)^2\overset{AM-GM}{\ge}\frac{1}{3}(3+3\lambda)^2=3(\lambda+1)^2$$

Equality holds for $a=b=c$.

1706. **If $a,b,c>0,\lambda\ge 0, ab+bc+ca\le 3abc$ then:**

$$\sum\frac{1}{1+\lambda a}\le\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$ab+bc+ca\le 3abc \text{ or}, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le 3\ (1)$$

$$\sum\frac{1}{1+\lambda a}=\sum\left(1-\left(\frac{\lambda a}{1+\lambda a}\right)\right)=3-\lambda\sum\frac{a}{1+\lambda a}=$$

$$=3-\lambda\sum\frac{1}{\frac{1}{a}+\lambda}\overset{CBS}{\le}3-\lambda\frac{(1+1+1)^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3\lambda}\overset{(1)}{\le}3-\frac{9\lambda}{3+3\lambda}=3-\frac{3\lambda}{1+\lambda}=\frac{3}{\lambda+1}$$

Equality holds for $a=b=c=1$.

1707. If $a,b,c>0$ then:

$$\sum\frac{(a^2+\lambda bc)(b^2+\lambda ac)}{a^2b^2}\ge 3(1+\lambda)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{(a^2+\lambda bc)(b^2+\lambda ac)}{a^2b^2}=\frac{c^2}{ab}\left(\frac{a^2+\lambda bc}{bc}\right)\left(\frac{b^2+\lambda ac}{ac}\right)=$$

$$=\frac{c^2}{ab}\left(\frac{a^2}{bc}+\lambda\right)\left(\frac{b^2}{ac}+\lambda\right)\overset{C-S}{\ge}\frac{c^2}{ab}\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2$$

$$\sum\frac{(a^2+\lambda bc)(b^2+\lambda ac)}{a^2b^2}\ge\sum\frac{c^2}{ab}\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\prod\frac{c^2}{ab}\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2}=3\left(\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2\left(\frac{\sqrt{ab}}{c}+\lambda\right)^2\right)^{\frac{1}{3}}\overset{Holder}{\ge}$$

$$\ge 3\left(1^{\frac{1}{3}}+(\lambda^3)^{\frac{1}{3}}\right)^{3\times\frac{2}{3}}\ge 3(1+\lambda)^2$$

Equality holds for $a=b=c=1$.

1708. If $a,b,c>0$ then:

$$\sum a^4\frac{b+c}{bc}\ge 6abc$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \frac{1}{c}\ge\frac{1}{b}\ge\frac{1}{a}$$

$$\sum a^4\frac{b+c}{bc}=\sum a^4\left(\frac{1}{b}+\frac{1}{c}\right)\overset{Chebyshev}{\ge}\frac{1}{3}\left(\sum a^4\right)\left(\sum\left(\frac{1}{b}+\frac{1}{c}\right)\right)\ge$$

$$\ge\frac{1}{3}\left(\sum a^4\right)\cdot 2\sum\frac{1}{a}\overset{AM-GM}{\ge}\frac{2}{3}\cdot\ 3(abc)^{\frac{4}{3}}\cdot\ 3\left(\frac{1}{abc}\right)^{\frac{1}{3}}=3abc$$

Equality holds for $a=b=c=1$.

1709. If $a,b,c>0$ then:

$$\sum a^{3n+1}\frac{b+c}{bc}\ge 6a^nb^nc^n$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ \frac{1}{a}\le\frac{1}{b}\le\frac{1}{c}$$

$$\sum a^{3n+1}\frac{b+c}{bc}=\sum a^{3n+1}\left(\frac{1}{b}+\frac{1}{c}\right)\overset{Chebyshev}{\ge}$$

$$\ge\frac{1}{3}\left(\sum a^{3n+1}\right)\sum\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{3}\left(\sum a^{3n+1}\right)\cdot 2\sum\frac{1}{a}\overset{AM-GM}{\ge}$$

$$\ge\frac{2}{3}\,.\,3(abc)^{\frac{3n+1}{3}}.\frac{3}{(abc)^{\frac{1}{3}}}=6a^nb^nc^n$$

Equality case: $a=b=c=1$.

1710. If $a,b,c>0$ and $a^3+b^3+c^3+3abc=6$ then:

$$\sum\frac{a^5+2}{a^2}\ge 9$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a^3+b^3+c^3+3abc=6\ \ or\ 3abc+3abc\overset{AM-GM}{\le}6\ or\ abc\le 1$$

$$\sum\frac{a^5+2}{a^2}=\sum a^3+2\sum\frac{1}{a^2}\overset{a^3+b^3+c^3+3abc=6\ \&AM-GM}{\ge}$$

$$\ge\ 6-3abc+\frac{6}{abc}\overset{abc\le 1}{\ge}\ 6-3\cdot 1+\frac{6}{1}=9$$

Equality holds for $a=b=c=1$.

1711. If $x, y, z > 0$ *and* $(3x+1)(3y+1)(3z+1) \leq 8$ then:

$$\frac{1}{6x+1}+\frac{1}{6y+1}+\frac{1}{6z+1} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ 3x = a, 3y = b, 3z = c\,.$$

$$(3x+1)(3y+1)(3z+1) \leq 8\ or, (a+1)(b+1)(c+1) \leq 8$$

$$2\sqrt{a}\cdot 2\sqrt{b}\cdot 2\sqrt{c} \overset{AM-GM}{\leq} (a+1)(b+1)(c+1) \leq 8$$

$$\sqrt{abc} \leq 1\,, abc \leq 1$$

$$a+b+c \overset{AM-GM}{\geq} 3\sqrt[3]{abc} \geq 3abc\,, since\ 0 < abc \leq 1\ then\ \sqrt[3]{abc} \geq abc\ (1)$$

$$\sum\big((1+2a)(1+2b)\big) = \sum(1+2(a+b)+4ab) = 3+4\sum a+4\sum ab =$$

$$= 1+2\left(1+\sum a\right)+2\sum a+4\sum ab \overset{abc\leq 1\ \&\ AM-GM}{\geq}$$

$$\geq\ 1+2\big(abc+\ 3\sqrt[3]{abc}\big)+2\sum a+4\sum ab \geq$$

$$\overset{(1)}{\geq}\ 1+2(abc+\ 3abc)+2\sum a+4\sum ab = 1+8abc+2\sum a+4\sum ab\ (2)$$

$$\prod(1+2a) = 1+2\sum a+4\sum ab+8abc$$

$$\frac{1}{6x+1}+\frac{1}{6y+1}+\frac{1}{6z+1} = \sum\frac{1}{6x+1} = \sum\frac{1}{2a+1} =$$

$$= \frac{\sum\big((1+2a)(1+2b)\big)}{\prod(1+2a)} \overset{(2)}{\geq} \frac{1+8abc+2\sum a+4\sum ab}{1+8abc+2\sum a+4\sum ab} = 1$$

$$Equality\ holds\ for\ a=b=c=1\ or\ x=y=z=\frac{1}{3}.$$

1712. If $a, b, c > 0$ and $a^2 + b^2 + c^2 + abc = 4$, then prove that :

$$a(1 + c\sqrt{b}) + b(1 + a\sqrt{c}) + c(1 + b\sqrt{a}) \le 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2 + b^2 + c^2 + abc = 4 \Rightarrow a^2 + a.bc + (b^2 + c^2 - 4) = 0$$

$$\Rightarrow a = \frac{-bc \pm \sqrt{b^2c^2 - 4(b^2 + c^2 - 4)}}{2} = \frac{-bc \pm \sqrt{(4 - b^2)(4 - c^2)}}{2}$$

$$\therefore a = \frac{-bc + \sqrt{(4 - b^2)(4 - c^2)}}{2} \ (\because a > 0) \Rightarrow 2 - a$$

$$= \frac{8 + 2bc - 2\sqrt{(4 - b^2)(4 - c^2)}}{4} \overset{A-G}{\ge} \frac{8 + 2bc - (8 - b^2 - c^2)}{4} = \frac{(b + c)^2}{4}$$

$$\Rightarrow \sqrt{2 - a} \ge \frac{b + c}{2} \overset{A-G}{\Rightarrow} \frac{2 - a + 1}{2} \ge \frac{b + c}{2} \Rightarrow 3 \overset{(1)}{\ge} \sum_{cyc} a \text{ and LHS} =$$

$$\sum_{cyc} a + \sum_{cyc} (bc.\sqrt{a}) \overset{\text{Chebyshev}}{\le} \sum_{cyc} a + \frac{1}{3}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} \sqrt{a}\right) \overset{CBS}{\le}$$

$$\sum_{cyc} a + \frac{1}{9}\left(\sum_{cyc} a\right)^2 . \sqrt{3\sum_{cyc} a} \overset{\text{via (1)}}{\le} 3 + 3 = 6$$

$$\forall a, b, c > 0 \mid a^2 + b^2 + c^2 + abc = 4, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1713. If $a, b, c > \frac{1}{2}$, then prove that :

$$\frac{1}{2a - 1} + \frac{1}{2b - 1} + \frac{1}{2c - 1} + \frac{4ab}{ab + 1} + \frac{4bc}{bc + 1} + \frac{4ca}{ca + 1} \ge 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } 2a - 1 = x, 2b - 1 = y, 2c - 1 = z \text{ and then :}$$

$$\frac{1}{2}\left(\frac{1}{2a - 1} + \frac{1}{2b - 1}\right) + \frac{4ab}{ab + 1} - 3 = \frac{x + y}{2xy} + \frac{4(x + 1)(y + 1)}{(x + 1)(y + 1) + 4} - 3$$

$$= \frac{m^2 + 3mn + 5m + 2n^2 - 22n}{2xy\big((x + 1)(y + 1) + 4\big)} \overset{?}{\ge} 0 \ (m = x + y, n = xy)$$

$$\Leftrightarrow m^2+(3n+5)m+2n^2-22n\overset{?}{\geq}0$$

$$\Leftrightarrow m\overset{?}{\geq}\frac{-(3n+5)+\sqrt{(3n+5)^2-4(2n^2-22n)}}{2}\ \textit{and}\ \because m\overset{A-G}{\geq}2\sqrt{n}$$

$$\therefore \text{it suffices to prove}: 4p+3p^2+5\overset{?}{\geq}\sqrt{p^4+118p^2+25}\ (p=\sqrt{n})$$

$$\Leftrightarrow (4p+3p^2+5)^2-p^4-118p^2-25\overset{?}{\geq}0\Leftrightarrow 8p(p+5)(p-1)^2\overset{?}{\geq}0\to \text{true}$$

$$\therefore \frac{1}{2}\left(\frac{1}{2a-1}+\frac{1}{2b-1}\right)+\frac{4ab}{ab+1}\geq 3\ \textit{and analogs}$$

$$\therefore \frac{1}{2a-1}+\frac{1}{2b-1}+\frac{1}{2c-1}+\frac{4ab}{ab+1}+\frac{4bc}{bc+1}+\frac{4ca}{ca+1}\geq 9$$

$$\forall a,b,c>\frac{1}{2},''=''\ \text{iff}\ a=b=c=1\ (QED)$$

1714. If $a,b,c>0$ and $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then prove that:

$$(a+b+c)^2+18abc\geq 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a=\sum_{cyc}\frac{1}{a}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}a}\Rightarrow\sum_{cyc}a\geq 3\ \textit{and if}\ abc>1, \textit{then}:$$

$$\text{LHS}>9+18=27\ \textit{and we now focus on}: abc\leq 1\ \textit{and then}:$$

$$\sum_{cyc}a=\frac{\sum_{cyc}ab}{abc}\geq\sum_{cyc}ab\Rightarrow \text{LHS}\geq\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+18abc\overset{?}{\geq}27$$

$$\overset{\sum_{cyc}a=\frac{\sum_{cyc}ab}{abc}}{=}\left(\frac{27abc(\sum_{cyc}a)}{\sum_{cyc}ab}\right)^{\frac{3}{2}}$$

$$\Leftrightarrow\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+18abc\right)^2\ \boxed{\overset{?}{\geq}\ (*)}\ \frac{729\left(abc(\sum_{cyc}a)\right)^3}{(\sum_{cyc}ab)^3}\ \textit{and assigning}$$

$$b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$$

$$\textit{and}\ z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z\ \text{form sides}$$

of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s, R, r}$ (say);

$$\text{and then}: \sum_{cyc} a = s, abc = r^2 s \text{ and } \sum_{cyc} ab = 4Rr + r^2 \text{ and so, } (*) \Leftrightarrow$$

$$(s(4Rr + r^2) + 18r^2 s)^2 \overset{?}{\geq} \frac{729(r^2 s^2)^3}{(4Rr + r^2)^3} \Leftrightarrow (4R + 19r)^2 (4R + r)^3 \overset{?}{\underset{(**)}{\geq}} 729rs^4$$

$$\text{and } 729rs^4 \overset{\text{Gerretsen}}{\leq} 729r(4R^2 + 4Rr + 3r^2)^2 \overset{?}{\leq} \text{LHS of } (**)$$

$$\Leftrightarrow 256t^5 - 292t^4 + 1816t^3 - 2498t^2 - 3253t - 1550 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(256t^4 + 220t^3 + 2256t^2 + 2014t + 775) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \text{combining both cases, } (a+b+c)^2 + 18abc \geq 27$$

$$\forall a, b, c > 0 \,\Big|\, \sum_{cyc} a = \sum_{cyc} \frac{1}{a}, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$

1715. If $a, b, c > 0$ and $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, then prove that:

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+2ab} + \frac{1}{1+2bc} + \frac{1}{1+2ca} \geq 1 \Leftrightarrow 1 + \sum_{cyc} ab \geq 4a^2b^2c^2$$

$$\overset{\sum_{cyc} a = \frac{\sum_{cyc} ab}{abc}}{\Leftrightarrow} \left(\frac{abc(\sum_{cyc} a)}{\sum_{cyc} ab}\right)^3 + \left(\frac{abc(\sum_{cyc} a)}{\sum_{cyc} ab}\right)^2 \left(\sum_{cyc} ab\right) \overset{(*)}{\geq} 4a^2b^2c^2 \text{ and}$$

assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s, R, r}$ (say);

$$\text{and then}: \sum_{cyc} a = s, abc = r^2 s \text{ and } \sum_{cyc} ab = 4Rr + r^2 \text{ and so,}$$

$$(*) \Leftrightarrow s^4 + (4R + r)^2 s^2 \overset{(**)}{\geq} 4r(4R + r)^3$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} ((4R + r)^2 + 16Rr - 5r^2)s^2 \overset{\text{Gerretsen}}{\geq}$$

$$(16R^2+24Rr-4r^2)(16Rr-5r^2)\overset{?}{\geq}4r(4R+r)^3\Leftrightarrow 8r^2(14R-r)(R-2r)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq 1$$

$$\forall a,b,c>0\,\Big|\,a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},''='' \text{ iff } a=b=c=1\text{ (QED)}$$

1716. If $a,b,c\in[0,2]$ and $a+b+c=3$, then prove that :

$$a^3+b^3+c^3+abc\leq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Either } b\leq 2, c<2 \text{ or } c\leq 2, b<2 \therefore (b-2)(c-2)\geq 0$$

$$\overset{a+b+c=3}{\Rightarrow} bc\geq 2(3-a)-4\Rightarrow bc\overset{(1)}{\geq}2-2a$$

$$\text{Now, } a^3+b^3+c^3+abc\leq 9\overset{a+b+c=3}{\Leftrightarrow}3abc+3\left(9-3\sum_{\text{cyc}}ab\right)+abc\leq 9$$

$$\overset{a+b+c=3}{\Leftrightarrow}4abc+18-9\big(bc+a(3-a)\big)\leq 0\Leftrightarrow(9-4a).bc\geq 18-9a(3-a)$$

$$\Leftrightarrow bc\overset{(*)}{\geq}\frac{9(a-1)(a-2)}{9-4a}\left(\because 9-4a\overset{a\leq 2}{\geq}1>0\right)$$

$$\text{Now, let us assume that : } bc<\frac{9(a-1)(a-2)}{9-4a}\text{ and then, since } bc\geq 0$$

$$\therefore\frac{9(a-1)(a-2)}{9-4a}>0\Rightarrow a-1<0\left(\begin{matrix}\because a-2\leq 0\text{ and more specifically,}\\ \text{in this case : } a-2<0\end{matrix}\right)$$

$$\Rightarrow a<1\rightarrow(m)\text{ and also, via (1), }2-2a\leq bc<\frac{9(a-1)(a-2)}{9-4a}$$

$$\Leftrightarrow(a-1)\left(\frac{9(a-2)}{9-4a}+2\right)>0\Leftrightarrow\frac{a(a-1)}{9-4a}>0\Rightarrow a>1\rightarrow(n)$$

$(\because a\geq 0$ and more specifically, in this case : $a>0)$ and $(m),(n)\Rightarrow$
our assumption leads to a contradiction and hence, our assumption is incorrect

$$\Rightarrow(*)\text{ is true}\therefore a^3+b^3+c^3+abc\leq 9\ \forall a,b,c\in[0,2]\,\big|\,a+b+c=3,$$

$$''='' \text{ iff } (a=1,b=0,c=2)\text{ or }(a=1,b=2,c=0)\text{ or }(b=1,c=0,a=2)$$

$$\text{or }(b=1,c=2,a=0)\text{ or }(c=1,a=0,b=2)\text{ or }(c=1,a=2,b=0)\text{ (QED)}$$

1717. If $a, b, c > 0$ and $a^2+b^2+c^2=3$, then prove that :

$$2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\ge 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

and then : $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} ab = 4Rr+r^2, \sum_{cyc} a^2 = s^2-8Rr-2r^2$ and

$\sum_{cyc} a^2b^2 = r^2((4R+r)^2-2s^2)$ and so, $2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\ge 9$

$$\overset{3=a^2+b^2+c^2}{\Leftrightarrow} 2\left((s^2-8Rr-2r^2)^2-2r^2((4R+r)^2-2s^2)\right)(4Rr+r^2)+9r^4s^2$$

$$-(4Rr+r^2)(s^2-8Rr-2r^2)^2\ge 0 \Leftrightarrow (4R+r)s^2 \overset{(*)}{\ge} r(64R^2-13r^2)$$

Now, $(4R+r)s^2 \overset{Rouche}{\ge} (4R+r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$

$$\overset{?}{\ge} r(64R^2-13r^2)$$

$$\Leftrightarrow 2(R-2r)(4R^2-3Rr-3r^2)\overset{?}{\ge} 2(R-2r)(4R+r).\sqrt{R^2-2Rr}$$

and to prove it, it suffices to prove : $(4R^2-3Rr-3r^2)^2 \overset{?}{>} (R^2-2Rr)(4R+r)^2$

$\Leftrightarrow r^3(20R+9r)\overset{?}{>}0 \to$ true $\Rightarrow (*)$ is true$2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\ge 9$

$$\forall a, b, c>0 \,\middle|\, a^2+b^2+c^2=3, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$

1718. If $x, y \in [0,1]$, then prove that :

$$8(x+y-1)^2-9xy(x+y-1)+xy\ge 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

LHS is a quadratic polynomial in $(x+y-1)$ with discriminant,

$\delta = 81x^2y^2-32xy$ and if : $xy\le\frac{32}{81}$, then : $\delta\le 0\Rightarrow$ LHS ≥ 0 and we now

$$\text{focus on}: xy > \frac{32}{81} \text{ and, in order to prove}: LHS \geq 0, \text{it suffices to prove}:$$

$$x + y - 1 \overset{?}{\geq} \frac{9xy + \sqrt{81x^2y^2 - 32xy}}{16} \text{ and to prove it, it suffices to prove}:$$

$$32\sqrt{xy} - 16 - 9xy \underset{(*)}{\overset{?}{\geq}} \sqrt{81x^2y^2 - 32xy} \left(\because x + y \overset{A-G}{\geq} 2\sqrt{xy}\right)$$

$$\text{Now}, 9\mu^2 - 32\mu + 16 \overset{?}{<} 0 \left(\mu = \sqrt{xy}\right) \Leftrightarrow \mu < \frac{32 + \sqrt{448}}{18} \text{ and } \mu > \frac{32 - \sqrt{448}}{18}$$

$$\rightarrow \text{both are true} \because \mu \overset{x,y \in [0,1]}{\leq} 1 < \frac{32 + \sqrt{448}}{18} \text{ and } \mu \overset{xy > \frac{32}{81}}{>} \frac{4\sqrt{2}}{9} > \frac{32 - \sqrt{448}}{18}$$

$$\therefore 9xy - 32\sqrt{xy} + 16 < 0 \Rightarrow 32\sqrt{xy} - 16 - 9xy > 0 \text{ and so}, (*)$$

$$\Leftrightarrow (32\mu - 16 - 9\mu^2)^2 \overset{?}{\geq} 81\mu^4 - 32\mu^2 \Leftrightarrow 64(1 - \mu)(3\mu - 2)^2 \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\because x, y \in [0,1] \Rightarrow 1 \geq \mu \Rightarrow (*) \text{ is true} \therefore 8(x + y - 1)^2 - 9xy(x + y - 1) + xy \geq 0$$

$$\forall x, y \in [0,1], '' ='' \text{ iff } x = y = 1 \text{ (QED)}$$

1719. If $a, b, c > 0$ then:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{(a + b + c)^{2025}}{a^{2024}\sqrt{b} + b^{2024}\sqrt{c} + c^{2024}\sqrt{a}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \sum \frac{a}{b} = \sum \frac{a^{2025}}{ba^{2024}}$$

$$\left(\sum \frac{a^{2025}}{ba^{2024}}\right)\left(\sum a^{2024}\sqrt{b}\right)^{2024} \overset{Holder}{\geq}$$

$$\geq \left(\sum \sqrt[2025]{\frac{a^{2025}}{ba^{2024}} \cdot a^{2024}b}\right)^{2025} = (a + b + c)^{2025}$$

$$\left(\sum \frac{a^{2025}}{ba^{2024}}\right) \geq \frac{(a + b + c)^{2025}}{\left(\sum a^{2024}\sqrt{b}\right)^{2024}}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = \sum \frac{a}{b} = \sum \frac{a^{2025}}{ba^{2024}} =$$

$$=\frac{(a+b+c)^{2025}}{\left(\sum a\sqrt[2024]{b}\right)^{2024}}=\frac{(a+b+c)^{2025}}{a\sqrt[2024]{b}+b\sqrt[2024]{c}+c\sqrt[2024]{a}}$$

Equality holds for $a=b=c$.

1720. If $a,b,c>0$ then:

$$\frac{1}{(a+b)^3}+\frac{1}{(b+c)^3}+\frac{1}{(c+a)^3}\ge\frac{9(24abc)^4}{[(a+b)^3+(b+c)^3+(c+a)^3]^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{1}{(a+b)^3}+\frac{1}{(b+c)^3}+\frac{1}{(c+a)^3}=\sum\frac{1}{(a+b)^3}=$$

$$=\sum\frac{(a+b)^{12}}{(a+b)^{15}}=\sum\frac{((a+b)^2)^6}{((a+b)^3)^5}\overset{Radon}{\ge}\frac{(\sum(a+b)^2)^6}{(\sum(a+b)^3)^5}\ge$$

$$\overset{AM-GM}{\ge}\frac{(\sum 4ab)^6}{(\sum(a+b)^3)^5}\overset{AM-GM}{\ge}\frac{\left(12\,(a^2b^2c^2)^{\frac{1}{3}}\right)^6}{(\sum(a+b)^3)^5}=\frac{12^6(abc)^4}{(\sum(a+b)^3)^5}=$$

$$=9.\frac{(3.8.abc)^4}{(\sum(a+b)^3)^5}=\frac{9(24abc)^4}{[(a+b)^3+(b+c)^3+(c+a)^3]^5}$$

Equality holds for a=b=c.

1721. If $a,b,c>0\,,\ n\ge 2$ then:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{(a+b+c)^{n+1}}{\left(a\sqrt[n]{b+c}+b\sqrt[n]{c+a}+c\sqrt[n]{a+b}\right)^n}$$

Proposed by Zazazhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\sum\frac{a}{b+c}=\sum\frac{a^{n+1}}{a^n(b+c)}$$

$$\left(\sum\frac{a^{n+1}}{a^n(b+c)}\right)\left(\sum a\sqrt[n]{b+c}\right)^n\overset{Holder}{\ge}\left(\sum\sqrt[n+1]{\frac{a^{n+1}}{a^n(b+c)}.a^n(b+c)}\right)^{n+1}=(a+b+c)^{n+1}$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\sum\frac{a}{b+c}=\sum\frac{a^{n+1}}{a^n(b+c)}\geq$$

$$\geq\frac{(a+b+c)^{n+1}}{\left(\sum a\sqrt[n]{b+c}\right)^n}=\frac{(a+b+c)^{n+1}}{\left(a\sqrt[n]{b+c}+b\sqrt[n]{c+a}+c\sqrt[n]{a+b}\right)^n}$$

Equality holds for a=b=c.

1722. If $a,b,c\in\mathbb{R}$ then:

$$\sqrt[3]{ln(1+e^a)ln(1+e^b)ln(1+e^c)}\leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

Proposed by Daniel Sitaru-Romania

Solution by Christos Tsifakis-Greece

Let be $f(x)=ln\big(ln(1+e^x)\big), x\in\mathbb{R}$.

$$f'(x)=\frac{e^x}{(1+e^x)ln(1+e^x)}$$

It is known that:

$$ln(1+t)<t, t>-1\Longrightarrow ln(1+e^x)<1+e^x-1=e^x$$

$$ln(1+e^x)-e^x<0$$

$f''(x)=\frac{e^x(ln(1+e^x)-e^x)}{(1+e^x)^2ln^2(1+e^x)}<0\Longrightarrow f$ –concave. By Jensen's inequality:

$$\frac{f(a)+f(b)+f(c)}{3}\leq f\left(\frac{a+b+c}{3}\right)$$

$$\frac{1}{3}\left(ln\big(ln(1+e^a)\big)+ln\left(ln\big(1+e^b\big)\right)+ln\big(ln(1+e^c)\big)\right)\leq ln\left(ln\left(1+e^{\frac{a+b+c}{3}}\right)\right)$$

$$ln\sqrt[3]{ln(1+e^a)+ln(1+e^b)+ln(1+e^c)}\leq ln\left(ln\left(1+\sqrt[3]{e^{a+b+c}}\right)\right)$$

$$\sqrt[3]{ln(1+e^a)+ln(1+e^b)+ln(1+e^c)}\leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

$$\sqrt[3]{ln\big((1+e^a)(1+e^b)(1+e^c)\big)}\leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

Equality holds for $a=b=c$.

1723. If $a,b,c>0$ and $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then prove that :

$$(1-abc)\left(a^2+b^2+c^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right)\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

and then : $\sum_{cyc} a = s \to (1),, abc = r^2s \to (2), \sum_{cyc} ab = 4Rr+r^2 \to (3),$

$\sum_{cyc} a^2 = s^2-8Rr-2r^2 \to (4)$ and $\sum_{cyc} a^2b^2 = r^2((4R+r)^2-2s^2) \to (5)$ and so,

$$(1-abc)\left(\sum_{cyc} a^2-\sum_{cyc}\frac{1}{a^2}\right)\geq 0 \Leftrightarrow \sum_{cyc} a^2-\sum_{cyc}\frac{1}{a^2}\geq abc\sum_{cyc} a^2-abc\sum_{cyc}\frac{1}{a^2}$$

$$\overset{abc\sum_{cyc} a=\sum_{cyc} ab}{\Leftrightarrow}\left(\frac{1-a^2b^2c^2}{1+abc}\right)\left(\sum_{cyc} a^2-\frac{\sum_{cyc} a^2b^2}{a^2b^2c^2}\cdot\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^2\right)\geq 0$$

$$\overset{abc\sum_{cyc} a=\sum_{cyc} ab}{\Leftrightarrow}\frac{\left(\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^3-a^2b^2c^2\right)\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2-\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2\right)}{(1+abc)\left(\sum_{cyc} ab\right)^2}$$

$$\geq 0 \Leftrightarrow \frac{a^2b^2c^2}{(1+abc)\left(\sum_{cyc} ab\right)^5}\cdot\left(abc\left(\sum_{cyc} a\right)^3-\left(\sum_{cyc} ab\right)^3\right)\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2-\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2\right)\geq 0$$

$$\overset{\text{via }(1),(2),(3),(4)\text{ and }(5)}{\Leftrightarrow}\left(r^2s^4-r^3(4R+r)^3\right)\left((s^2-8Rr-2r^2).r^2(4R+r)^2-r^2s^2((4R+r)^2-2s^2)\right)\geq 0$$

$$\Leftrightarrow 2r^4(s^4-r(4R+r)^3)(s^4-r(4R+r)^3)\geq 0 \Leftrightarrow 2r^4(s^4-r(4R+r)^3)^2\geq 0$$

$$\to \text{true} \therefore (1-abc)\left(a^2+b^2+c^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right)\geq 0$$

$$\forall\, a,b,c>0 \,\Big|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \text{''}=\text{''} \text{ iff } a=b=c=1 \text{ (QED)}$$

1724. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$a^2+b^2+c^2\geq a^2b^2c^2(a^3+b^3+c^3)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$$

$$y+z-x=2a>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$$

$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$=s,R,r\ (say);\text{then}:\sum_{cyc}a=s\to(1), abc=r^2s\to(2),$$

$$\sum_{cyc}a^2=s^2-8Rr-2r^2\to(3)\text{ and }\sum_{cyc}a^3=s(s^2-12Rr)\to(4)$$

$$\text{and so, } a^2+b^2+c^2\geq a^2b^2c^2(a^3+b^3+c^3)\overset{a+b+c=3}{\Leftrightarrow}$$

$$\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)^7\geq 2187a^2b^2c^2\left(\sum_{cyc}a^3\right)$$

$$\overset{\text{via }(1),(2),(3)\text{ and }(4)}{\Leftrightarrow}(s^2-8Rr-2r^2)s^7\geq 2187r^4s^3(s^2-12Rr)$$

$$\Leftrightarrow s^4(s^2-8Rr-2r^2)\ \boxed{\overset{(*)}{\geq}}\ 2187r^4(s^2-12Rr)$$

$$\text{and}\because P=(s^2-16Rr+5r^2)^3+(40Rr-17r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (*),\text{ it suffices to prove : LHS of } (*)\geq P$$

$$\Leftrightarrow(128R^2-116Rr-523r^2)s^2\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ r(1536R^3-1728R^2r-5931Rr^2-75r^3)$$

$$\boxed{\text{Case 1}}\ 128R^2-116Rr-523r^2\geq 0\text{ and then : LHS of }(**)\overset{\text{Gerretsen}}{\geq}$$

$$(128R^2-116Rr-523r^2)(16Rr-5r^2)\overset{?}{\geq}r\begin{pmatrix}1536R^3-1728R^2r-5931Rr^2\\-75r^3\end{pmatrix}$$

$$\Leftrightarrow 512t^3-768t^2-1857t+2690\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\big((t-2)(512t+1280)+1215\big)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true}$$

Case 2 $128R^2 - 116Rr - 523r^2 < 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(128R^2 - 116Rr - 523r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} r\begin{pmatrix}1536R^3 - 1728R^2r \\ -5931Rr^2 - 75r^3\end{pmatrix}$$

$$\Leftrightarrow 512t^4 - 1488t^3 - 444t^2 + 3491t - 1494 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(512t^2 + 560t - 252) + 243\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true } \forall \Delta XYZ$$

$$\therefore a^2 + b^2 + c^2 \geq a^2b^2c^2(a^3 + b^3 + c^3)\ \forall a,b,c > 0 \,\Big|\, \sum_{cyc} a = 3,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1725. If $a,b,c > 0$, then prove that :

$$a^3 + b^3 + c^3 + 6abc \geq 9 \cdot \frac{a^2 + b^2 + c^2}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0,$$

$$y + z - x = 2a > 0 \text{ and } z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$$

$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$= s, R, r \text{ (say); and then : } abc = r^2s \to (1), \sum_{cyc} ab = 4Rr + r^2 \to (2),$$

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3) \text{ and } \sum_{cyc} a^3 = s(s^2 - 12Rr) \to (4) \text{ and so,}$$

$$a^3 + b^3 + c^3 + 6abc \geq 9.\frac{a^2 + b^2 + c^2}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}$$

$$(s(s^2 - 12Rr) + 6r^2s)(4Rr + r^2) \geq 9r^2s(s^2 - 8Rr - 2r^2)$$

$$\Leftrightarrow (R - 2r)s^2 \geq r(R - 2r)(12R + 3r)$$

$$\Leftrightarrow (R - 2r)\left((s^2 - 16Rr + 5r^2) + 4r(R - 2r)\right) \geq 0 \to \text{true via Gerretsen and Euler}$$

$$\therefore a^3+b^3+c^3+6abc \geq 9.\frac{a^2+b^2+c^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \forall a,b,c>0,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1726. If $a,b,c>0$, then prove that :

$$\frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)}$$
$$=\sum_{cyc}\frac{a}{\left(\frac{a}{a+b}+1\right)(b+c)}=\sum_{cyc}\frac{a^2}{ab+ac+\frac{a^2(b+c)}{a+b}}$$
$$\overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a^2}{a+b}\right)-\sum_{cyc}\frac{a^3}{a+b}} \overset{?}{\geq} 1$$
$$\Leftrightarrow \left(\sum_{cyc}a\right)^2+\sum_{cyc}\frac{a^3}{a+b} \overset{?}{\geq} 2\sum_{cyc}ab+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a(a+b-b)}{a+b}\right)$$
$$=2\sum_{cyc}ab+\left(\sum_{cyc}a\right)^2-\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right)$$
$$\Leftrightarrow \sum_{cyc}\frac{a^3}{a+b}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right) \boxed{\overset{?}{\geq}(*)} 2\sum_{cyc}ab$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc}a=s \rightarrow (1), abc=r^2s \rightarrow (2)$,

$$\sum_{cyc}ab=4Rr+r^2 \rightarrow (3), \sum_{cyc}a^2=s^2-8Rr-2r^2 \rightarrow (4)$$

$$\text{Now, } \sum_{cyc}\frac{a^3}{a+b}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right)$$

$$=\sum_{cyc}\frac{a^4}{a^2+ab}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a^2b^2}{a^2b+ab^2}\right)\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2+\sum_{cyc}ab}+\frac{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)^2}{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}\overset{?}{\geq}2\sum_{cyc}ab$$

$$\Leftrightarrow\frac{\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)^2}{\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)}$$

$$\overset{?}{\geq}2\sum_{cyc}ab\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}(s(4Rr+r^2)-3r^2s)(s^2-8Rr-2r^2)^2$$

$$+s(s^2-4Rr-r^2)(4Rr+r^2)^2\overset{?}{\geq}2(4Rr+r^2)(s(4Rr+r^2)-3r^2s)(s^2-4Rr-r^2)$$

$$\Leftrightarrow(4R-2r)s^4-r(80R^2-32Rr-13r^2)s^2$$

$$+r^2(320R^3-48R^2r-84Rr^2-13r^3)\boxed{\underset{(**)}{\overset{?}{\geq}}}0\ \&\ \because P=(4R-2r)(s^2-16Rr+5r^2)^2$$

$$+r(48R^2-72Rr+33r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove }(**),$$

it suffices to prove : LHS of $(**)\overset{?}{\geq}P\Leftrightarrow 2t^3-9t^2+12t-4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$

$$\Leftrightarrow(2t-1)(t-2)^2\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)}\geq 1\ \forall a,b,c>0,$$

$$''=''\text{ iff }a=b=c\text{ (QED)}$$

1727. If $x,y,z>0$ then:

$$\prod(x^2y^2+2)\prod\left(\frac{1}{(x+y)^4}+1\right)>\frac{729}{64}$$

Proposed by D.M.Bătinețu-Giurgiu, Mihaly Bencze-Romania

Solution by Tapas Das-India

$$\prod(x^2y^2+2)=\prod\left((xy)^2+(\sqrt{2})^2\right)\overset{\text{ARKADY ALT'S inequality}}{\geq}$$

$$\geq\frac{3}{4}(\sqrt{2})^4(xy+yz+zx)^2=3(xy+yz+zx)^2$$

$$\prod\left(\frac{1}{(x+y)^4}+1\right)=\prod\left(\left(\frac{1}{(x+y)^2}\right)^2+1^2\right)\overset{\text{ARKADY ALT'S inequality}}{\geq}$$

$$\geq \frac{3}{4}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2$$

$$\prod(x^2y^2+2)\prod\left(\frac{1}{(x+y)^4}+1\right)\geq$$

$$\geq 3(xy+yz+zx)^2.\frac{3}{4}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^2=$$

$$=\frac{9}{4}\left[(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\right]^2 \overset{Iran\ 1996}{\geq} \frac{9}{4}.\left(\frac{9}{4}\right)^2>\frac{729}{64}$$

1728. If $a, b, c > 0$ and $\lambda \leq 6$, then :

$$\sum_{cyc} a^3+\lambda abc \geq (\lambda+3)\frac{\sum_{cyc} a^2}{\sum_{cyc}\frac{1}{a}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,

$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$

$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$= s, R, r$ (say); & then : $\sum_{cyc} a = s \to (1), abc = r^2 s \to (2), \sum_{cyc} ab = 4Rr + r^2 \to (3)$,

$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$ & $\sum_{cyc} a^3 = s(s^2-12Rr) \to (5)$ and

$$\sum_{cyc} a^3+\lambda abc-(\lambda+3)\frac{\sum_{cyc} a^2}{\sum_{cyc}\frac{1}{a}}$$

$$=\sum_{cyc} a^3 - 3abc\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}-\frac{\lambda abc}{\sum_{cyc} ab}\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$\overset{\lambda\leq 6}{\geq}\sum_{cyc} a^3 - 3abc\frac{\sum_{cyc} a^2}{\sum_{cyc} ab}-\frac{6abc}{\sum_{cyc} ab}\left(\sum_{cyc} a^2-\sum_{cyc} ab\right) \overset{via\ (2),(3),(4)\ and\ (5)}{=}$$

$$\frac{s(s^2-12Rr)(4Rr+r^2)-3r^2s(s^2-8Rr-2r^2)-6r^2s(s^2-12Rr-3r^2)}{4Rr+r^2}$$

$$=\frac{4s\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right)}{4R+r}$$

$$\overset{\text{Gerretsen}}{\geq}\frac{4s\left((R-2r)(16Rr-5r^2)-r(12R^2-21Rr-6r^2)\right)}{4R+r}=\frac{16rs(R-2r)^2}{4R+r}$$

$$\overset{\text{Euler}}{\geq}0\therefore\sum_{\text{cyc}}a^3+\lambda abc\geq(\lambda+3)\frac{\sum_{\text{cyc}}a^2}{\sum_{\text{cyc}}\frac{1}{a}}\forall a,b,c>0\wedge\lambda\leq 6,$$

$$''='' \text{ iff } a=b=c \text{ (QED)}$$

1729. If $a,b,c\in[0,1]$ and $a+b+c=2$, then prove that :

$$a^4+b^4+c^4+\frac{11}{2}abc\leq\frac{5}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^4+b^4+c^4+\frac{11}{2}abc-\frac{5}{2}$$

$$=a^4+(b+c)^4-2bc(2(b+c)^2-bc)+\frac{11}{2}abc-\frac{5}{2}\overset{?}{\leq}0$$

$$\overset{a+b+c=2}{\Leftrightarrow}a^4+(2-a)^4-4bc(2-a)^2+2b^2c^2+\frac{11}{2}abc-\frac{5}{2}\overset{?}{\leq}0$$

$$\Leftrightarrow 4b^2c^2+(11a-8(2-a)^2)bc+2a^4+2(2-a)^4-5\underset{(1)}{\overset{?}{\leq}}0$$

Now, LHS of (1) is a quadratic polynomial in bc with discriminant $\delta=$

$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5)$$

$$=\frac{(4509-2160a)(5a-4)^2+1856-1080a}{125}\overset{a\leq 1}{>}0$$

and so, in order to prove : (1), it suffices to prove :

$$bc\overset{(i)}{\leq}\frac{8a^2-43a+32+\sqrt{\delta}}{8}\text{ AND }bc\overset{(ii)}{\geq}\frac{8a^2-43a+32-\sqrt{\delta}}{8}$$

$$\because(b-1)(c-1)\geq 0\overset{a+b+c=2}{\therefore}bc\geq 2-a-1\therefore\text{ in order to prove (ii),}$$

$$\text{it suffices to prove : } 8-8a \overset{?}{\geq} 8a^2-43a+32-\sqrt{\delta}$$

$$\Leftrightarrow \sqrt{\delta} \underset{(*)}{\overset{?}{\geq}} 8a^2-35a+24 \text{ and it's trivially true when : } 8a^2-43a+32 \leq 0$$

$$\text{and so, we now focus on the case when : } 8a^2-43a+32>0 \text{ and then : } (*) \Leftrightarrow$$

$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5) \overset{?}{\geq} (8a^2-35a+24)^2$$

$$\Leftrightarrow 16(2a-1)^2\big(1+a(1-a)\big) \overset{?}{\geq} 0 \to \text{true} \because 0 \leq a \leq 1 \Rightarrow (*) \Rightarrow \text{(ii) is true}$$

$$\text{and again, via AM}-\text{GM, bc} \leq \frac{(2-a)^2}{4} \overset{?}{\leq} \frac{8a^2-43a+32+\sqrt{\delta}}{8}$$

$$\Leftrightarrow 6a^2-35a+24+\sqrt{\delta} \underset{(**)}{\overset{?}{\geq}} 0 \text{ and it's trivially true when : } 6a^2-35a+24 \geq 0$$

$$\text{and so, we now focus on the case when : } 6a^2-35a+24<0 \text{ and then : } (**) \Leftrightarrow$$

$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5) \overset{?}{\geq} (6a^2-35a+24)^2$$

$$\Leftrightarrow 4(1-a)(9a^3+8a^2+4(a-1)^2) \overset{?}{\geq} 0 \to \text{true} \because 0 \leq a \leq 1 \Rightarrow (**) \Rightarrow \text{(i) is true}$$

$$\text{and so, both (i) and (ii) are true} \Rightarrow (1) \text{ is true}$$

$$\therefore a^4+b^4+c^4+\frac{11}{2}abc \leq \frac{5}{2} \ \forall a,b,c \in [0,1] \,\big|\, a+b+c=2,$$

$$''='' \text{ iff } \left(a=\frac{1}{2}, b=1, c=\frac{1}{2}\right) \text{ or } \left(a=\frac{1}{2}, b=\frac{1}{2}, c=1\right) \text{ or } \left(b=\frac{1}{2}, c=1, a=\frac{1}{2}\right)$$

$$\text{or } \left(b=\frac{1}{2}, c=\frac{1}{2}, a=1\right) \text{ or } \left(c=\frac{1}{2}, a=1, b=\frac{1}{2}\right) \text{ or } \left(c=\frac{1}{2}, a=\frac{1}{2}, b=1\right)$$

(QED)

1730. If $a,b,c \in [0,1]$ and $a+b+c=2$, then prove that :

$$a^6+b^6+c^6+2abc \leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$b^6+c^6=(b+c)^6-6bc(b^4+c^4)-b^2c^2(15b^2+15c^2+20bc)$$
$$=(b+c)^6-6bc\Big((b+c)^4-2bc(2(b+c)^2-bc)\Big)-b^2c^2(15(b+c)^2-10bc)$$
$$\overset{a+b+c=2}{=} (2-a)^6-6bc(2-a)^4+9b^2c^2(2-a)^2-2b^3c^3$$

$$\leq (2-a)^6 - 6bc(2-a)^4 + 9b^2c^2(2-a)^2 - 2b^2c^2(1-a)$$

$$\left(\because (b-1)(c-1) \geq 0 \overset{a+b+c=2}{\Rightarrow} bc \geq 2-a-1\right) \Rightarrow a^6+b^6+c^6+2abc-2 \boxed{\leq} E$$

$$= (9(2-a)^2 - 2 + 2a)b^2c^2 + (2a - 6(2-a)^4)bc + a^6 + (2-a)^6 - 2 \rightarrow (1)$$

Now, E is a quadratic polynomial in bc with discriminant $\delta =$
$(2a - 6(2-a)^4)^2 - 4(9(2-a)^2 - 2 + 2a)(a^6 + (2-a)^6 - 2)$ and $\because$ $P =$
$-36a(a-1)^7 - 124(a-1)^7 - 144a(a-1)^5 + 24a(a-1)^4 + 408(a-1)^4$
$-144(a-1)^3 - 116a(a-1) \geq 0 \therefore$ in order to prove $\delta > 0$,
it suffices to prove : $\delta > P \Leftrightarrow 108 - 92a > 0 \rightarrow$ true $\because a \leq 1 \therefore \delta > 0$ and so,
in order to prove : $E \leq 0$, it suffices to prove :

$$bc \overset{(i)}{\boxed{\leq}} \frac{6(2-a)^4 - 2a + \sqrt{\delta}}{2(9(2-a)^2 - 2 + 2a)} \text{ AND } bc \overset{(ii)}{\boxed{\geq}} \frac{6(2-a)^4 - 2a - \sqrt{\delta}}{2(9(2-a)^2 - 2 + 2a)}$$

$\because bc \geq 1 - a$ and $\because 9(2-a)^2 - 2 + 2a \overset{a \leq 1}{\geq} 7 > 0 \therefore$ in order to prove (ii),
it suffices to prove : $2(1-a)(9(2-a)^2 - 2 + 2a) \overset{?}{\geq} 6(2-a)^4 - 2a - \sqrt{\delta}$

$$\Leftrightarrow \sqrt{\delta} \overset{?}{\geq} 2(a-2)(3a^3 - 9a^2 + 11a - 7)$$

$$\Leftrightarrow (2a - 6(2-a)^4)^2 - 4(9(2-a)^2 - 2 + 2a)(a^6 + (2-a)^6 - 2)$$

$$\overset{?}{\geq} 4(a-2)^2(3a^3 - 9a^2 + 11a - 7)^2$$

$$\left(\begin{array}{c}\because 3a^3 - 9a^2 + 11a - 7 = (a-1)(3(a-1)^2 + 2) - 2 \overset{a \leq 1}{\leq} -2 < 0 \\ \overset{a \leq 1 < 2}{\Rightarrow} (a-2)(3a^3 - 9a^2 + 11a - 7) > 0\end{array}\right)$$

$$\Leftrightarrow 4a(1-a)\left(18a^6 - 104a^5 + 303a^4 - 591a^3 + 716a^2 - 442a + 136\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 18a^6 - 104a^5 + 303a^4 - 591a^3 + 716a^2 - 442a + 136 \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because$$

$Q = 18(a-1)^6 + 4a(a-1)^4 + 49(a-1)^4 - 59(a-1)^3 \geq 0 \therefore$ in order
to prove $(*)$, it suffices to prove : LHS of $(*) \overset{?}{\geq} Q \Leftrightarrow 9(1-a^2) + 1 + 35a \overset{?}{\geq} 0$
$\rightarrow$ true(strict inequality) $\Rightarrow (*) \Rightarrow$ (ii) is true and again, via AM – GM,

$$bc \leq \frac{(2-a)^2}{4} \overset{?}{\leq} \frac{6(2-a)^4 - 2a + \sqrt{\delta}}{2(9(2-a)^2 - 2 + 2a)}$$

$$\Leftrightarrow 24(2-a)^4 - 8a - 2(9(2-a)^2 - 2 + 2a)(2-a)^2 + 4\sqrt{\delta} \overset{?}{\geq} 0$$

$$\Leftrightarrow 3a^4 - 26a^3 + 82a^2 - 116a + 56 + 2\sqrt{\delta} \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and it's trivially true when :}$$

$3a^4 - 26a^3 + 82a^2 - 116a + 56 \geq 0$ and so, we now focus on
the case when : $3a^4 - 26a^3 + 82a^2 - 116a + 56 < 0$ and then : $(**) \Leftrightarrow$

$$2\sqrt{\delta} \overset{?}{\geq} -(3a^4 - 26a^3 + 82a^2 - 116a + 56) \Leftrightarrow$$

$$4(2a - 6(2-a)^4)^2 - 16(9(2-a)^2 - 2 + 2a)(a^6 + (2-a)^6 - 2)$$

$$\overset{?}{\geq} (3a^4 - 26a^3 + 82a^2 - 116a + 56)^2$$

$$\Leftrightarrow a\begin{pmatrix}(a-1)^4.\left((1-a)(153a^2+97a+251)+745\right)+(a+186)(a-1)^2\\+63+931(1-a)\end{pmatrix}\overset{?}{\geq}0$$

$\rightarrow$ true $\therefore 0\leq a\leq 1\Rightarrow(**)\Rightarrow$ (i) is true and so, both (i) and (ii) are true $\Rightarrow E\leq 0$

$\overset{\text{via }(1)}{\Rightarrow} a^6+b^6+c^6+2abc-2\leq 0 \therefore a^6+b^6+c^6+2abc\leq 2\ \forall\, a,b,c\in[0,1]$ such that $a+b+c=2$, $''=''$ iff $(a=0,b=1,c=1)$ or $(b=0,c=1,a=1)$ or $(c=0,a=1,b=1)$ (QED)

1731. If $x,y,z\geq 0$ and $x+y+z=3$, then prove that :

$$5(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z})+9\geq 8(xy+yz+zx)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Firstly $\forall\, x\geq 0$, we have : $5\sqrt[5]{x}+3x\geq 8\sqrt{x}\rightarrow$ (i) and $\because$ LHS = RHS = 0 for $x=0$, so we now focus on : $x>0$and then : LHS of (i) =

$$\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+x+x+x\overset{A-G}{\geq}8\sqrt[8]{x.x^3}=8\sqrt{x}\Rightarrow \text{(i) is true}$$

$\forall\, x\geq 0$ and analogously, $5\sqrt[5]{y}+3y\geq 8\sqrt{y}$ and $5\sqrt[5]{z}+3z\geq 8\sqrt{z}$

$$\therefore 5(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z})+9\overset{x+y+z=3}{=}5(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z})+3(x+y+z)$$

$\geq 8\sum_{cyc}\sqrt{x}$ and so, it suffices to prove : $\sum_{cyc}a\overset{(*)}{\geq}\sum_{cyc}a^2b^2$ subject to $\sum_{cyc}a^2=3$ $(a=\sqrt{x},b=\sqrt{y},c=\sqrt{z})$ and $(*)$ is trivially true when exactly 2 variables equal to zero and we now focus on the case when exactly 1 variable equals to zero and WLOG we may assume $a=0$ $(b,c>0\mid b^2+c^2=3)$ and then : $(*)\Leftrightarrow$

$$b+c\geq b^2c^2\overset{b^2+c^2=3}{\Leftrightarrow}(b^2+c^2).\sqrt{b^2+c^2}.(b+c)\geq b^2c^2.3\sqrt{3}\rightarrow\text{true}$$

(strict inequality) $\because$ LHS $\overset{A-G}{\geq}4\sqrt{2}.b^2c^2>3\sqrt{3}.b^2c^2\Rightarrow(*)$ is true and we now consider the case when : $a,b,c>0$ and assigning $b+c=X,c+a=Y,a+b=Z$ $\Rightarrow X+Y-Z=2c>0, Y+Z-X=2a>0$ and $Z+X-Y=2b>0$ $\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle XYZ with semiperimeter, circumradius & inradius = s, R, r (say); and then :

$\sum_{cyc}a=s\rightarrow(1), \sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(2)$ and $\sum_{cyc}a^2b^2=$ $r^2((4R+r)^2-2s^2)\rightarrow(3)$ and (1), (2), (3) together with $\sum_{cyc}a^2=3\Rightarrow(*)\Leftrightarrow$

$$(s^2-8Rr-2r^2).\sqrt{s^2-8Rr-2r^2}.s\geq 3\sqrt{3}.r^2((4R+r)^2-2s^2)$$

$$\Leftrightarrow s^2(s^2-8Rr-2r^2)^3-27r^4((4R+r)^2-2s^2)^2\overset{(**)}{\geq}0 \text{ and } \because P=$$

$$(s^2-16Rr+5r^2)^4+2r(20R-13r)(s^2-16Rr+5r^2)^3$$
$$+2r^2(288R^2-396Rr+72r^2)(s^2-16Rr+5r^2)^2$$
$$+2r^3(1792R^3-2976R^2r+1308Rr^2+55r^3)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\overset{?}{\geq}P$

$$\Leftrightarrow 160t^4-416t^3+60t^2+361t-194\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(160t^2+224t+316)+729\right)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)$$

$$\Rightarrow (*)\text{ is true and combining all cases, }(*)\text{ is true }\forall a,b,c\geq 0\,\Big|\,\sum_{cyc}a^2=3$$

$$\therefore 5\left(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}\right)+9\geq 8(xy+yz+zx)\ \forall x,y,z\geq 0\,|\,x+y+z=3,$$
$$''=''\ \text{iff } x=y=z=1\ (\text{QED})$$

1732. If $x,y,z>0$ and $xyz=1$, then prove that :

$$\frac{yz}{\sqrt{x+y}}+\frac{xz}{\sqrt{y+z}}+\frac{xy}{\sqrt{z+x}}\geq\frac{3\sqrt{2}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{yz}{\sqrt{x+y}}+\frac{xz}{\sqrt{y+z}}+\frac{xy}{\sqrt{z+x}}=\sum_{cyc}\frac{(xz)^{\frac{3}{2}}}{\sqrt{xyz+z^2x}}\overset{\text{Radon}}{\geq}$$

$$\frac{\left(\sum_{cyc}xy\right)^{\frac{3}{2}}}{\sqrt{3xyz+\sum_{cyc}x^2y}}\overset{\text{CBS}}{\geq}\frac{\left(\sum_{cyc}xy\right)^{\frac{3}{2}}}{\sqrt{3xyz+\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)}}}\overset{?}{\geq}\frac{3\sqrt{2}}{2}$$

$$\overset{xyz=1}{\Leftrightarrow}2\left(\sum_{cyc}xy\right)^3-27x^2y^2z^2\overset{?}{\geq}9xyz.\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)}$$

$$\Leftrightarrow\left(2\left(\sum_{cyc}xy\right)^3-27x^2y^2z^2\right)^2\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 81x^2y^2z^2\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0,$
$b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b$
$\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r\ (say) \Rightarrow \sum_{cyc} x = s, xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2, \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2$$
$$and \sum_{cyc} x^2y^2 = r^2((4R+r)^2 - 2s^2) and\ so, (*)$$
$$\Leftrightarrow (2(4Rr + r^2)^3 - 27r^4s^2)^2 - 81r^4s^2.r^2((4R+r)^2 - 2s^2)(s^2 - 8Rr - 2r^2) \overset{?}{\underset{(**)}{\geq}} 0$$
$$and \because P = 162r^6(s^2 - 16Rr + 5r^2)(s^2 - 4R^2 - 4Rr - 3r^2)^2$$
$$+2r^6(972Rr + 243r^2)(s^2 - 16Rr + 5r^2)^2$$
$$-4r^6(648R^4 + 5616R^3r - 11016R^2r^2 + 4050Rr^3 + 351r^4)\begin{pmatrix} s^2 - 4R^2 - 4Rr \\ -3r^2 \end{pmatrix}$$
$$\overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} P$$
$$\Leftrightarrow 376t^6 - 2136t^5 + 10248t^4 - 23170t^3 + 17448t^2 - 1317t - 1478 \overset{?}{\geq} 0$$
$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)\begin{pmatrix} 376t^3(t-2) + 120t^3 + 6216t^2 \\ +4222t + 9472 \end{pmatrix} + 19683\right) \overset{?}{\geq} 0$$
$$\rightarrow true \because t \overset{Euler}{\geq} 2 \Rightarrow (**) \Rightarrow (*) is\ true \therefore \frac{yz}{\sqrt{x+y}} + \frac{xz}{\sqrt{y+z}} + \frac{xy}{\sqrt{z+x}} \geq \frac{3\sqrt{2}}{2}$$
$$\forall x, y, z > 0 \mid xyz = 1, '' = '' \text{ iff } x = y = z = 1\ (QED)$$

1733. If $a, b, c \in [0, 2]$ and $a + b + c = 3$, then prove that :

$$a^4 + b^4 + c^4 \leq 17$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Either } b \leq 2, c < 2 \text{ or } c \leq 2, b < 2 \therefore (b-2)(c-2) \geq 0$$
$$\overset{a+b+c=3}{\Rightarrow} bc \geq 2(3-a) - 4 \Rightarrow bc \overset{(1)}{\geq} 2 - 2a$$
$$Now, a^4 + b^4 + c^4 \leq 17 \Leftrightarrow a^4 + (b+c)^4 - 2bc(2(b+c)^2 - bc) \leq 17$$
$$\overset{a+b+c=3}{\Leftrightarrow} 2b^2c^2 - 4bc(3-a)^2 + a^4 + (3-a)^4 - 17 \leq 0$$
$$\Leftrightarrow bc \overset{(m)}{\leq} (3-a)^2 + \sqrt{-6a^3 + 27a^2 - 54a + 49}\ and$$
$$bc \overset{(n)}{\geq} (3-a)^2 - \sqrt{-6a^3 + 27a^2 - 54a + 49}$$
$$\begin{pmatrix} \because 16(3-a)^4 - 8(a^4 + (3-a)^4 - 17) = 16(-6a^3 + 27a^2 - 54a + 49) \\ = 16\left((2-a)(6a^2 - 15a + 24) + 1\right) > 0 \therefore \Delta_{6a^2-15a+24} < 0\ and\ 2 \geq a \end{pmatrix}$$
$$Now, bc \overset{A-G}{\leq} \frac{(3-a)^2}{4} < (3-a)^2 + \sqrt{-6a^3 + 27a^2 - 54a + 49}$$
$$\Rightarrow (m) \text{ is true and to prove } (n), \text{ we segregate into 2 cases :}$$

$$\boxed{\text{Case 1}}\ a \le 1 \text{ and then : } bc \overset{\text{via }(1)}{\ge} 2-2a \overset{?}{\ge}$$
$$(3-a)^2 - \sqrt{-6a^3+27a^2-54a+49}$$
$$\Leftrightarrow \sqrt{-6a^3+27a^2-54a+49} \overset{?}{\ge} a^2-4a+7 \Leftrightarrow$$
$$-6a^3+27a^2-54a+49 \overset{?}{\ge} (a^2-4a+7)^2 \left(\because \Delta_{a^2-4a+7} < 0 \Rightarrow a^2-4a+7>0\right)$$
$$\Leftrightarrow a(1-a)\left((1-a)+1+a^2\right) \overset{?}{\ge} 0 \to \text{true} \therefore a \le 1 \Rightarrow (\text{n}) \text{ is true}$$
$$\boxed{\text{Case 2}}\ a > 1 \text{ and then : } bc \ge 0 \overset{?}{\ge} (3-a)^2 - \sqrt{-6a^3+27a^2-54a+49}$$
$$\Leftrightarrow -6a^3+27a^2-54a+49 \overset{?}{\ge} (3-a)^4 \Leftrightarrow (a-1)(2-a)(a^2-3a+16) \overset{?}{\ge} 0$$
$$\to \text{true} \because \Delta_{a^2-3a+16} < 0 \Rightarrow a^2-3a+16>0 \text{ and } 1<a\le 2 \Rightarrow (\text{n}) \text{ is true}$$
$$\therefore \text{combining both cases, (n) and (m) are true } \forall a,b,c \in [0,2] \,\middle|\, a+b+c=3$$
$$\therefore a^4+b^4+c^4 \le 17\ \forall a,b,c \in [0,2] \,\middle|\, a+b+c=3$$
$$''='' \text{ iff } (a=0,b=1,c=2) \text{ or } (a=0,b=2,c=1) \text{ or } (b=0,c=1,a=2)$$
$$\text{or } (b=0,c=2,a=1) \text{ or } (c=0,a=1,b=2) \text{ or } (c=0,a=2,b=1) (\text{QED})$$

1734. If $a,b,c>1$ and $2(a+b+c)=ab+bc+ca$, then prove that :

$$\frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}} \ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}} = \sum_{cyc}\frac{\sqrt{3}.a}{\sqrt{(b+1)(3b-3)}} \overset{A-G}{\ge} \sum_{cyc}\frac{\sqrt{3}.a}{\frac{b+1+3b-3}{2}}$$
$$= \sqrt{3}.\sum_{cyc}\frac{a}{2b-1} = \sqrt{3}.\sum_{cyc}\frac{a^2}{2ab-a} \overset{\text{Bergstrom}}{\ge} \frac{\sqrt{3}.\left(\sum_{cyc} a\right)^2}{2\sum_{cyc} ab - \sum_{cyc} a} \overset{2(a+b+c)=ab+bc+ca}{=}$$
$$\frac{\sqrt{3}.\left(\sum_{cyc} a\right)^2}{4\sum_{cyc} a - \sum_{cyc} a} = \frac{\sum_{cyc} a}{\sqrt{3}} \ge \frac{6}{\sqrt{3}} \left(\because 2\sum_{cyc} a = \sum_{cyc} ab \le \frac{\left(\sum_{cyc} a\right)^2}{3} \Rightarrow \sum_{cyc} a \ge 6\right)$$
$$\therefore \frac{a}{\sqrt{b^2-1}}+\frac{b}{\sqrt{c^2-1}}+\frac{c}{\sqrt{a^2-1}} \ge 2\sqrt{3}$$
$$\forall a,b,c>1 \,\middle|\, 2(a+b+c)=ab+bc+ca, ''='' \text{ iff } a=b=c=2 \ (\text{QED})$$

1735. If $a,b>0$ and $ab \le 1$, then prove that :

$$\frac{1}{a^3+b}+\frac{1}{b^3+a} \le \frac{1}{a+b}+\frac{4}{(a+b)^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3+b}+\frac{1}{b^3+a}=\frac{a^3+b^3+a+b}{a^3b^3+ab+a^4+b^4}\overset{A-G}{\leq}\frac{a^3+b^3+a+b}{2a^2b^2+a^4+b^4}$$

$$=\frac{(a+b)^3-3ab(a+b)+a+b}{(a^2+b^2)^2}=\frac{x^3-3xy+x}{(x^2-2y)^2}\ (x=a+b \text{ and } y=ab)$$

$$\overset{?}{\leq}\frac{1}{a+b}+\frac{4}{(a+b)^3}=\frac{4+x^2}{x^3}\Leftrightarrow(3-y)x^4+(4y^2-16y)x^2+16y^2\overset{?}{\underset{(*)}{\geq}}0$$

and $\because$ LHS of $(*)$ is a quadratic polynomial in x^2 with discriminant $\delta=(4y^2-16y)^2-64y^2(3-y)=16y^2(2-y)^2\ \therefore$ in order to prove $(*)$,

$$\text{it suffices to prove}: x^2\overset{?}{\geq}\frac{16y-4y^2+4y(2-y)}{2(3-y)}=\frac{24y-8y^2}{2(3-y)}$$

$$(\because (3-y),(2-y)>0 \text{ as } y\leq 1)\ \to \text{true} \because 2x^2(3-y)\overset{A-G}{\geq}8y(3-y)=24y-8y^2$$

$$\Rightarrow (*) \text{ is true}\ \therefore \frac{1}{a^3+b}+\frac{1}{b^3+a}\leq\frac{1}{a+b}+\frac{4}{(a+b)^3}\ \forall a,b>0 \mid ab\leq 1,$$

$$''='' \text{ iff } a=b=1\ (QED)$$

1736. If $0<a,b,c<\frac{3}{2}$ and $abc=(3-2a)(3-2b)(3-2c)$

then prove that : $abc\leq 1$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let us assume that : $abc>1$ and then :

$$\sqrt[2]{(3-2a)(3-2b)(3-2c)}>1\overset{\text{via } A-G}{\Rightarrow}\frac{1}{3}\sum_{cyc}(3-2a)>1$$

$$\Rightarrow 9-3>2\sum_{cyc}a\Rightarrow 3>\sum_{cyc}a\overset{A-G}{\geq}3\sqrt[3]{abc}\Rightarrow abc<1,$$

a contradiction $\Rightarrow$ our assumption is incorrect $\therefore abc\leq 1$

$$\forall a,b,c\in\left(0,\frac{3}{2}\right) \text{ and } abc=(3-2a)(3-2b)(3-2c)\ (QED)$$

1737. If $a,b,c\geq -2$ and $a+b+c=3$, then prove that :

$$a^2+b^2+c^2+abc\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a+2=x, b+2=y, c+2=z \text{ and then}: a^2+b^2+c^2+abc-4$$

$$=\sum_{cyc}(x-2)^2+\prod_{cyc}(x-2)-4$$

$$=\sum_{cyc}x^2-4\sum_{cyc}x+12+xyz-2\sum_{cyc}xy+4\sum_{cyc}x-8-4\overset{x+y+z=9}{=}$$

$$\frac{1}{9}\left(\sum_{cyc}x^2-2\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)+xyz=\frac{1}{9}\left(\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\right)$$

$$\overset{\text{Schur}}{\geq}0\ (\because x,y,z\geq 0)\ \therefore a^2+b^2+c^2+abc\geq 4\forall\ a,b,c\geq -2\ \big|\ a+b+c=3,$$

$$''='' \text{ iff } (a=b=c=1) \text{ or } \left(a=-2, b=c=\frac{5}{2}\right) \text{ or } \left(b=-2, c=a=\frac{5}{2}\right)$$

$$\text{or } \left(c=-2, a=b=\frac{5}{2}\right) \text{ (QED)}$$

1738. If $a,b,c>0$ and $a+b+c=3$, then prove that:

$$\frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$$

$$y+z-x=2a>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$$

$\Rightarrow x,y,z$ **form sides of a triangle XYZ with semiperimeter, circumradius and inradius**

$$=s,R,r\ (say);\ \text{and then}: \sum_{cyc}a=s\rightarrow(1), abc=r^2s\rightarrow(2),$$

$$\sum_{cyc}ab=4Rr+r^2\rightarrow(3), \sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4)$$

$$\text{Now, }\frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)}=\sum_{cyc}\frac{(a^2+abc)^2}{a^2b^2+2abc^2}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} a^2+3abc\right)^2}{\sum_{cyc} a^2b^2+2abc\sum_{cyc} a} \overset{a+b+c=3}{=} \frac{\left(\sum_{cyc} a^2+\frac{9abc}{\sum_{cyc} a}\right)^2}{\left(\sum_{cyc} ab\right)^2} \overset{\text{via (1),(2),(3) and (4)}}{=}$$

$$\frac{(s^2-8Rr-2r^2+9r^2)^2}{(4Rr+r^2)^2} \overset{?}{\geq} 4 \Leftrightarrow s^2-8Rr+7r^2 \overset{?}{\geq} 8Rr+2r^2 \Leftrightarrow s^2 \overset{?}{\geq} 16Rr-5r^2$$

$$\rightarrow \text{true via Gerretsen} \therefore \frac{a(a+bc)^2}{b(ab+2c^2)}+\frac{b(b+ca)^2}{c(bc+2a^2)}+\frac{c(c+ab)^2}{a(ca+2b^2)} \geq 4$$

$$\forall a,b,c>0 \,\big|\, a+b+c=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1739. If $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\lambda \geq 0$, then :

$$\sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}} \geq \frac{3}{\sqrt{\lambda+3}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}} = \sum_{cyc}\frac{a^2}{\sqrt{\lambda a+3abc}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc}\sqrt{\lambda a+3abc}} \overset{\text{CBS}}{\geq}$$

$$\frac{\left(\sum_{cyc} a\right)^2}{\sqrt{3\lambda\sum_{cyc} a+27abc}} \overset{?}{\geq} \frac{3}{\sqrt{\lambda+3}}$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a\right)^4+3\left(\sum_{cyc} a\right)^4 \underset{(*)}{\overset{?}{\geq}} 27\lambda\left(\sum_{cyc} a\right)+243abc$$

$$\text{Now, } \sum_{cyc} ab = abc\sum_{cyc} a \leq \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \Rightarrow \sum_{cyc} ab \geq 3 \Rightarrow \left(\sum_{cyc} a\right)^2 \geq 3\sum_{cyc} ab \geq 9$$

$$\Rightarrow \sum_{cyc} a \geq 3 \text{ and so, since } \lambda \geq 0, \lambda\left(\sum_{cyc} a\right)^4+3\left(\sum_{cyc} a\right)^4 \geq$$

$$\lambda\left(\sum_{cyc} a\right).27+9\left(\sum_{cyc} a\right)^3 \overset{A-G}{\geq} 27\lambda\left(\sum_{cyc} a\right)+243abc \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}} \ge \frac{3}{\sqrt{\lambda+3}}\ \forall a,b,c>0 \,\middle|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$
$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1740. If $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ then :

$$\sum_{cyc}\sqrt{\frac{a^2+2}{3}} \le a+b+c$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{\frac{a^2+2}{3}} \overset{CBS}{\le} \sqrt{3}.\sqrt{\sum_{cyc}\frac{a^2+2}{3}} = \sqrt{\sum_{cyc}a^2+6} \overset{?}{\le} \sum_{cyc}a$$

$$\Leftrightarrow \sum_{cyc}a^2+6 \overset{?}{\le} \sum_{cyc}a^2+2\sum_{cyc}ab \Leftrightarrow \sum_{cyc}ab \overset{?}{\ge} 3 \to \text{true} \because \sum_{cyc}ab = abc\sum_{cyc}a \le$$

$$\frac{1}{3}\left(\sum_{cyc}ab\right)^2 \Rightarrow \sum_{cyc}ab \ge 3 \therefore \sum_{cyc}\sqrt{\frac{a^2+2}{3}} \le a+b+c$$

$$\forall a,b,c>0 \,\middle|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, '' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1741. If $x,y,t>0$ and $\lambda \ge 1$, then :

$$\frac{x^2+ty}{t+\lambda y}+\frac{y^2+tx}{t+\lambda x} \ge \frac{x+y}{\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^2+ty}{t+\lambda y}+\frac{y^2+tx}{t+\lambda x} \ge \frac{x+y}{\lambda} \overset{\text{simplifying}}{\Leftrightarrow}$$

$$\lambda^2(x^3+y^3-x^2y-xy^2+2txy)+\lambda(t^2x+t^2y-2txy)-t^2x-t^2y \ge 0$$

$$\Leftrightarrow \lambda^2(x-y)^2+2\lambda txy(\lambda-1)+t^2x(\lambda-1)+t^2y(\lambda-1) \ge 0 \to \text{true} \because x,y,t>0$$

$$and\ \lambda \geq 1 \therefore \frac{x^2+ty}{t+\lambda y}+\frac{y^2+tx}{t+\lambda x} \geq \frac{x+y}{\lambda}\ \forall\ x,y,t>0\ and\ \lambda \geq 1,$$

$$''='' \text{ iff } x=y \text{ and } \lambda=1\ (QED)$$

1742. If $a,b,c \geq 1$, then :

$$\sum_{cyc}\frac{1}{1+a} \geq \frac{9}{3+\sum_{cyc}\sqrt{ab}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\sqrt{a}, y=\sqrt{b}, z=\sqrt{c} \text{ and then : } \sum_{cyc}\frac{1}{1+a} \geq \frac{9}{3+\sum_{cyc}\sqrt{ab}}$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{1+x^2} \geq \frac{9}{3+\sum_{cyc}xy} \Leftrightarrow \frac{3+2\sum_{cyc}x^2+\sum_{cyc}x^2y^2}{1+x^2y^2z^2+\sum_{cyc}x^2+\sum_{cyc}x^2y^2} \geq \frac{9}{3+\sum_{cyc}xy}$$

$$\Leftrightarrow 3\sum_{cyc}xy+2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2y^2\right) \overset{(*)}{\geq}$$

$$3\sum_{cyc}x^2+6\sum_{cyc}x^2y^2+9x^2y^2z^2 \text{ and assigning } x=u+1, y=v+1, z=w+1$$

$$(u,v,w \geq 0) \text{ and subsequently, expanding and re-arranging, } (*) \Leftrightarrow$$

$$\sum_{cyc}u^3v^3+uvw\left(\sum_{cyc}u^2v+\sum_{cyc}uv^2\right)-9u^2v^2w^2$$

$$+4\left(\sum_{cyc}u^3v^2+\sum_{cyc}u^2v^3+uvw\sum_{cyc}u^2-3uvw\sum_{cyc}uv\right)$$

$$+8\left(\sum_{cyc}u^3v+\sum_{cyc}uv^3-2uvw\sum_{cyc}u\right)+8\left(\sum_{cyc}u^3-3uvw\right)$$

$$+4\left(\sum_{cyc}u^2-\sum_{cyc}uv\right) \overset{(**)}{\geq} 0$$

$$\text{Now, LHS of } (**) \geq 3u^2v^2w^2+\left(\sum_{cyc}uv\right)\left(\sum_{cyc}u^2v^2-uvw\sum_{cyc}u\right)$$

$$+uvw\sum_{cyc}\left(w(u^2+v^2-2uv)\right)-3u^2v^2w^2$$

$$+4\left(\sum_{cyc}\left(w^3(u^2+v^2)\right)-2uvw\sum_{cyc}uv+uvw\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)\right)$$

$$+8\left(\sum_{cyc}\left(uv(u^2+v^2)\right)-2uvw\sum_{cyc}u\right)$$

$$+8\left(3uvw+\left(\sum_{cyc}u\right)\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)-3uvw\right)$$

$$\geq uvw\sum_{cyc}(w(u-v)^2)+8uvw\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)+16\left(\sum_{cyc}u^2v^2-uvw\sum_{cyc}u\right)$$

$$\geq 0\left(\because\sum_{cyc}u^2v^2\geq uvw\sum_{cyc}u,\sum_{cyc}u^2\geq\sum_{cyc}uv\ and\ u^2+v^2\geq 2uv\ and\ analogs\ \forall u,v,w\in\mathbb{R}\ and\ \therefore u,v,w\geq 0\right)$$

$$\Rightarrow (**)\Rightarrow (*)\text{ is true }\therefore\sum_{cyc}\frac{1}{1+a}\geq\frac{9}{3+\sum_{cyc}\sqrt{ab}}\ \forall a,b,c\geq 1,$$

$$''=''\text{ iff } a=b=c=1\ (QED)$$

1743. If $a,b,c>0, abc=1$ and $\lambda\geq 0, n\in\mathbb{N}$, then:

$$\sum_{cyc}\frac{bc}{a^{n+1}(b+\lambda c)}\geq\frac{3}{\lambda+1}\left(\frac{\sum_{cyc}bc}{3}\right)^n$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{bc}{a^{n+1}(b+\lambda c)}\overset{abc=1}{=}\sum_{cyc}\frac{(bc)^{n+2}}{3^n(b+\lambda c)}\overset{Holder}{\geq}\frac{\left(\sum_{cyc}bc\right)^{n+2}}{3^n(\lambda+1)\left(\sum_{cyc}a\right)}=$$

$$\geq\frac{\left(\sum_{cyc}bc\right)^n.\left(3abc\sum_{cyc}a\right)}{3^n(\lambda+1)\left(\sum_{cyc}a\right)}\overset{abc=1}{=}\frac{3}{\lambda+1}\left(\frac{\sum_{cyc}bc}{3}\right)^n,''=''\text{ iff } a=b=c=1\ (QED)$$

1744. If $x, y, z > 0, x+y+z=1$ *and* $\lambda \le \frac{3}{2}$ then :

$$\sum_{cyc}(x^3+yz)+\lambda xyz \ge \frac{\lambda+12}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(x^3+yz)+\lambda xyz-\frac{\lambda+12}{27} \overset{x+y+z=1}{=}$$

$$\sum_{cyc}x^3+\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\frac{4}{9}\left(\sum_{cyc}x\right)^3-\frac{\lambda}{27}\left(\left(\sum_{cyc}x\right)^3-27xyz\right)$$

$$\overset{\lambda\le\frac{3}{2}}{\ge}\sum_{cyc}x^3+\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\frac{4}{9}\left(\sum_{cyc}x\right)^3-\frac{\left(\frac{3}{2}\right)}{27}\left(\left(\sum_{cyc}x\right)^3-27xyz\right)$$

$$=\frac{1}{18}\left(\sum_{cyc}x^3+3xyz-\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\right)\overset{\text{Schur}}{\ge}0$$

$$\therefore \sum_{cyc}(x^3+yz)+\lambda xyz \ge \frac{\lambda+12}{27}\ \forall x,y,z>0\,|\,x+y+z=1 \text{ and } \lambda\le\frac{3}{2},$$

$$''='' \text{ iff } x=y=z=\frac{1}{3}\ (\text{QED})$$

1745. If $x, y, z > 0$, $xyz = 1$ *and* $n \in N$, $n \ge 2$ then:

$$\sum\frac{x^n}{x+y+1}\ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$3=3.(1)^{\frac{1}{3}}\overset{xyz=1}{=}3(xyz)^{\frac{1}{3}}\overset{AM-GM}{\le}x+y+z\ (1)$$

$$\left(\sum\frac{x^n}{x+y+1}\right)\left(\sum(x+y+1)\right)(1+1+1)^{n-2}\overset{Holder}{\ge}\frac{(x+y+z)^n}{3^{n-2}(\sum(x+y+1))}=$$

$$=\frac{(x+y+z)^n}{3^{n-2}\left(3+2(x+y+z)\right)}\overset{(1)}{\geq}\frac{(x+y+z)^n}{3^{n-2}\left((x+y+z)+2(x+y+z)\right)}=$$

$$=\frac{(x+y+z)^{n-1}}{3^{n-1}}\overset{AM-GM}{\geq}\frac{3^{n-1}(xyz)^{\frac{n-1}{3}}}{3^{n-1}}=1\ (as\ xyz=1)$$

Equality holds for $x=y=z=1$.

1746. If $x,y,z>0, xyz=1,\ n\in N, n\geq 2, \lambda\geq 0$ **then:**

$$\sum\frac{x^n}{x+y+\lambda}\geq\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$3\lambda=\lambda\cdot 3\cdot(1)^{\frac{1}{3}}\overset{xyz=1}{=}\lambda\cdot 3(xyz)^{\frac{1}{3}}\overset{AM-GM}{\leq}\lambda(x+y+z)\ (1)$$

$$\left(\sum\frac{x^n}{x+y+\lambda}\right)\left(\sum(x+y+\lambda)\right)(1+1+1)^{n-2}\overset{Holder}{\geq}\frac{(x+y+z)^n}{3^{n-2}\left(\sum(x+y+\lambda)\right)}=$$

$$=\frac{(x+y+z)^n}{3^{n-2}\left(3\lambda+2(x+y+z)\right)}\overset{(1)}{\geq}\frac{(x+y+z)^n}{3^{n-2}\left(\lambda(x+y+z)+2(x+y+z)\right)}=$$

$$=\frac{(x+y+z)^{n-1}}{3^{n-2}(\lambda+2)}\overset{AM-GM}{\geq}\frac{3^{n-1}(xyz)^{\frac{n-1}{3}}}{3^{n-2}(\lambda+2)}=\frac{3}{\lambda+2}\ (as\ xyz=1)$$

Equality holds for $x=y=z$.

1747. If $a,b,c>0,\ \lambda\geq 2, abc=1$ **then:**

$$\sum\frac{1}{a+\lambda}\leq\frac{3}{3+\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum ab\overset{AM-GM}{\geq}3(abc)^{\frac{2}{3}}\overset{abc=1}{=}3\ (1)$$

$$\sum a\overset{AM-GM}{\geq}3(abc)^{\frac{1}{3}}\overset{abc=1}{=}3\ \ (2)$$

$$\sum \frac{1}{a+\lambda} = \frac{(a+\lambda)(b+\lambda)+(b+\lambda)(c+\lambda)+(c+\lambda)(a+\lambda)}{(a+\lambda)(b+\lambda)(c+\lambda)} =$$

$$= \frac{3\lambda^2 + 2\lambda(a+b+c) + ab+bc+ca}{\lambda^3 + \lambda^2(a+b+c) + \lambda(ab+bc+ca) + abc} \overset{abc=1}{=}$$

$$= \frac{3\lambda^2 + 2\lambda(a+b+c) + ab+bc+ca}{\lambda^3 + \lambda^2(a+b+c) + \lambda(ab+bc+ca) + 1}$$

We need to show:

$$\frac{3\lambda^2 + 2\lambda(a+b+c) + ab+bc+ca}{\lambda^3 + \lambda^2(a+b+c) + \lambda(ab+bc+ca) + 1} \le 1$$

$$\lambda^3 + \lambda^2(a+b+c) + \lambda(ab+bc+ca) + 1 \ge 3\lambda^2 + 2\lambda(a+b+c) + ab+bc+ca$$

$$\lambda^3 + (\lambda^2 - 2\lambda)(a+b+c) + (\lambda-1)(ab+bc+ca) + 1 - 3\lambda^2 \ge 0$$

$$\lambda^3 + (\lambda^2-2\lambda).3 + (\lambda-1)3 + 1 - 3\lambda^2 \overset{(1)\&(2)}{\ge} 0$$

$$\lambda^3 - 3\lambda - 2 \ge 0$$

$$(\lambda-2)(\lambda^2+2\lambda+1) \ge 0 \ \textit{true as}\ \lambda \ge 2$$

Equality holds for a=b=c=1.

1748. If $a,b,c > 0$ then:

$$abc\sum\frac{1}{a+b} \le \frac{1}{2}\sum ab$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$abc\sum\frac{1}{a+b} \overset{AM-HM}{\le} abc\frac{1}{4}\sum\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{1}{4}\sum(bc+ac) = \frac{1}{2}\sum ab$$

Equality holds for a=b=c.

1749. If $a,b,c > 0$, *then prove that* :

$$\sum_{cyc}\frac{5a}{\sqrt{27(b^2+c^2)+21bc}} \ge \sqrt{3}$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{5a}{\sqrt{27(b^2+c^2)+21bc}}=\sum_{cyc}\frac{5a^2}{\sqrt{a}.\sqrt{27a(b^2+c^2)+21abc}}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{5\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\left(\sqrt{a}.\sqrt{27a(b^2+c^2)+21abc}\right)}\overset{\text{CBS}}{\geq}\frac{5\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{27\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-18abc}}\overset{?}{\geq}\sqrt{3}$$

$$\Leftrightarrow 25\left(\sum_{cyc}a\right)^3\overset{?}{\geq}81\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-54abc$$

$$\Leftrightarrow 25\sum_{cyc}a^3\overset{?}{\geq}39abc+6\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\text{ and }\because 39abc\overset{\text{AM-GM}}{\leq}13\sum_{cyc}a^3$$

$$\therefore\text{ it suffices to prove : }12\sum_{cyc}a^3\overset{?}{\geq}6\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)$$

$$\Leftrightarrow 2\sum_{cyc}a^3\overset{?}{\geq}\sum_{cyc}a^2b+\sum_{cyc}ab^2\to\text{true via Schur and AM-GM}$$

$$\therefore\sum_{cyc}\frac{5a}{\sqrt{27(b^2+c^2)+21bc}}\geq\sqrt{3}\ \forall\, a,b,c>0,''=''\text{ iff }a=b=c\ (QED)$$

1750.

If $a,b,c\geq 0$ and $2(ab+bc+ca)+abc=32$ then prove that :

$$ab+bc+ca\leq 2(a+b+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum_{cyc}\left((b+4)(c+4)\right)-\prod_{cyc}(a+4)=32-abc-2\sum_{cyc}ab=0$$

$$\Rightarrow 2\sum_{cyc}\left((b+4)(c+4)\right)=\prod_{cyc}(a+4)\Rightarrow\sum_{cyc}\frac{1}{a+4}=\frac{1}{2}\to(1)\text{ and }\because\frac{1}{a+4}\overset{a\geq 0}{\leq}\frac{1}{4}$$

$$\therefore\text{ we can set : }\frac{1}{a+4}=\frac{1}{4}-x,\frac{1}{b+4}=\frac{1}{4}-y,\frac{1}{c+4}=\frac{1}{4}-z\ (x,y,z\geq 0)$$

$$\Rightarrow x+y+z=\frac{3}{4}-\sum_{cyc}\frac{1}{a+4}\overset{\text{via (1)}}{=}\frac{1}{4}\therefore\sum_{cyc}x=\frac{1}{4}\to(2)\text{ and }\because\frac{1}{a+4}=\frac{1-4x}{4}$$

$$\therefore a = \frac{4}{1-4x} - 4 = \frac{4x}{\frac{1}{4} - x} \overset{\text{via }(2)}{=} \frac{4x}{y+z} \text{ and so. } a = \frac{4x}{y+z}, b = \frac{4y}{z+x}, c = \frac{4z}{x+y}$$

$$\text{and hence, } ab + bc + ca \le 2(a+b+c) \text{ becomes : } \sum_{\text{cyc}} \frac{16xy}{(y+z)(z+x)} \le \sum_{\text{cyc}} \frac{8x}{y+z}$$

$$\Leftrightarrow \sum_{\text{cyc}} \left(x \left(x^2 + \sum_{\text{cyc}} xy \right) \right) \ge 2 \sum_{\text{cyc}} (xy(x+y))$$

$$\Leftrightarrow \sum_{\text{cyc}} x^3 + \sum_{\text{cyc}} x^2y + \sum_{\text{cyc}} xy^2 + 3xyz \ge 2 \left(\sum_{\text{cyc}} x^2y + \sum_{\text{cyc}} xy^2 \right)$$

$$\Leftrightarrow \sum_{\text{cyc}} x^3 + 3xyz \ge \sum_{\text{cyc}} x^2y + \sum_{\text{cyc}} xy^2 \rightarrow \text{true via Schur}$$

$$\therefore ab + bc + ca \le 2(a+b+c) \ \forall\, a,b,c \ge 0 \,\big|\, 2(ab+bc+ca) + abc = 32,$$

$$'' ='' \text{ iff } (a=b=c=2) \text{ or } (a=0, b=c=4) \text{ or } (b=0, c=a=4)$$

$$\text{or } (c=0, a=b=4) \text{ (QED)}$$

1751. If $a, b, c > 0$ then prove that :

$$\frac{\sqrt{bc}}{a + \sqrt{(a+b)(a+c)}} + \frac{\sqrt{ca}}{b + \sqrt{(b+c)(b+a)}} + \frac{\sqrt{ab}}{c + \sqrt{(c+a)(c+b)}} \ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{bc}}{a + \sqrt{(a+b)(a+c)}} + \frac{\sqrt{ca}}{b + \sqrt{(b+c)(b+a)}} + \frac{\sqrt{ab}}{c + \sqrt{(c+a)(c+b)}}$$

$$\overset{\text{CBS}}{\ge} \sum_{\text{cyc}} \frac{\sqrt{bc}}{\sqrt{a+a+b}.\sqrt{a+a+c}} = \sum_{\text{cyc}} \frac{(bc)^{\frac{3}{2}}}{\sqrt{bc(2ac+bc)(2ab+bc)}}$$

$$\overset{\text{Radon}}{\ge} \frac{\left(\sum_{\text{cyc}} ab\right)^{\frac{3}{2}}}{\sqrt{\sum_{\text{cyc}} (bc(2ac+bc)(2ab+bc))}} \overset{?}{\ge} 1$$

$$\Leftrightarrow\left(\sum_{cyc} ab\right)^3 \overset{?}{\geq} \sum_{cyc}\left(bc(2ac+bc)(2ab+bc)\right)$$

$$\Leftrightarrow abc\left(\sum_{cyc} a^2b+\sum_{cyc} ab^2-6abc\right)\overset{?}{\geq} 0 \rightarrow \text{true} \because \sum_{cyc} a^2b, \sum_{cyc} ab^2 \overset{AM-GM}{\geq} 3abc$$

$$\therefore \frac{\sqrt{bc}}{a+\sqrt{(a+b)(a+c)}}+\frac{\sqrt{ca}}{b+\sqrt{(b+c)(b+a)}}+\frac{\sqrt{ab}}{c+\sqrt{(c+a)(c+b)}} \geq 1$$

$$\forall a,b,c>0, " = " \text{ iff } a=b=c \text{ (QED)}$$

1752. If $a,b,c>0$ and $a+b+c=1$, then prove that :

$$\sqrt{2-(ab+bc+ca)+3abc} \geq 2\sqrt[3]{(a+b)(b+c)(c+a)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); and then : $\sum_{cyc} a = s \rightarrow (1), abc = r^2s \rightarrow (2)$,

$$\sum_{cyc} ab = 4Rr+r^2 \rightarrow (3), \prod_{cyc}(b+c) = 4Rrs \rightarrow (4)$$

$$\text{Now, } \sqrt{2-(ab+bc+ca)+3abc} \geq 2\sqrt[3]{(a+b)(b+c)(c+a)} \overset{a+b+c=1}{\Leftrightarrow}$$

$$\left(2\left(\sum_{cyc} a\right)^2-\sum_{cyc} ab+\frac{3abc}{\sum_{cyc} a}\right)^3 \geq 64\prod_{cyc}(b+c)^2$$

$$\overset{\text{via (1),(2),(3) and (4)}}{\Leftrightarrow}\left(2s^2-4Rr-r^2+\frac{3r^2s}{s}\right)^3 \geq 64.16R^2r^2s^2$$

$$\Leftrightarrow (s^2-2Rr+r^2)^3-128R^2r^2s^2 \overset{(*)}{\geq} 0 \text{ and } \because P = s^4(s^2-16Rr+5r^2)+ (10Rr-2r^2)s^2(s^2-16Rr+5r^2)+r^2(44R^2-94Rr+13r^2)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\geq} 0 \left(\because 44R^2-94Rr+13r^2 = (R-2r)(44R-6r)+r^2 \overset{\text{Euler}}{\geq} r^2>0\right)$$

$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove : LHS of } (*) \overset{?}{\geq} P \Leftrightarrow$$

$$(t-2)(87t^2-40t+4) \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{2-(ab+bc+ca)+3abc} \ge 2\sqrt[3]{(a+b)(b+c)(c+a)}$$

$$\forall a,b,c>0 \,\middle|\, a+b+c=1, ''='' \text{ iff } a=b=c=\frac{1}{3} \text{ (QED)}$$

1753. If $a,b,c>0,\ abc(a+b+c)\ge 3$ then:

$$a^{2026}+b^{2026}+c^{2026}\ge a^{2025}+b^{2025}+c^{2025}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$abc(a+b+c)\ge 3$$

$$(ab).(ac)+(ab)(bc)+(bc)(ac)\ge 3$$

$$\frac{(ab+bc+ac)^2}{3}\ge 3,\qquad (ab+bc+ac)^2\ge 9$$

$$\sum ab\ge 3$$

$$\frac{(\sum a)^2}{3}\ge 3$$

$$\sum a\ge 3\ (1)$$

$$a^{2026}+b^{2026}+c^{2026}=a^{2025+1}+b^{2025+1}+c^{2025+1}\overset{Chebyshev}{\ge}$$

$$\ge\frac{1}{3}(a+b+c)\left(a^{2025}+b^{2025}+c^{2025}\right)\overset{(1)}{\ge}a^{2025}+b^{2025}+c^{2025}$$

Equality holds for a=b=c=1.

1754. If $a,b,c>0,\ a+b+c\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ then:

$$\frac{a^2}{\sqrt{b+8}}+\frac{b^2}{\sqrt{c+8}}+\frac{c^2}{\sqrt{a+8}}\ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\overset{C-S}{\ge}9 \text{ or } (a+b+c)^2\overset{a+b+c\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\ge}9 \text{ or}$$

$$a+b+c\ge 3\ (1)$$

$$24=8\cdot 3\overset{(1)}{\leq} 8(a+b+c)\ (2)$$

$$\frac{a^2}{\sqrt{b+8}}+\frac{b^2}{\sqrt{c+8}}+\frac{c^2}{\sqrt{a+8}}\overset{CBS}{\geq}\frac{(a+b+c)^2}{\sqrt{b+8}+\sqrt{c+8}+\sqrt{a+8}}\overset{CBS}{\geq}$$

$$\geq\frac{(a+b+c)^2}{\sqrt{3(a+b+c+24)}}\overset{(2)}{\geq}\frac{(a+b+c)^2}{\sqrt{3\big((a+b+c+8(a+b+c)\big)}}=\frac{(a+b+c)^{\frac{3}{2}}}{\sqrt{27}}\overset{(1)}{\geq}\frac{(3)^{\frac{3}{2}}}{(3)^{\frac{3}{2}}}=1$$

Equality holds for a=b=c=1.

1755. If $a,b,c>0$ and $n\in\mathbb{N}$ then prove that :

$$\frac{a+b}{(b+c)^n}+\frac{b+c}{(c+a)^n}+\frac{c+a}{(a+b)^n}\geq\frac{(a+b+c)^{n+1}}{2^{n-1}(a^2+b^2+c^2)^n}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{AM-GM}{\geq}\frac{3}{\left(\sqrt[3]{\prod_{cyc}(b+c)}\right)^{n-1}}\overset{AM-GM}{\geq}\frac{3}{\left(\frac{2\sum_{cyc}a}{3}\right)^{n-1}}=\frac{3^n\left(\sum_{cyc}a\right)^{n+1}}{2^{n-1}\left(\left(\sum_{cyc}a\right)^2\right)^n}=$$

$$\overset{CBS}{\geq}\frac{3^n\left(\sum_{cyc}a\right)^{n+1}}{2^{n-1}\left(3\sum_{cyc}a^2\right)^n}=\frac{(a+b+c)^{n+1}}{2^{n-1}(a^2+b^2+c^2)^n},''='' \text{ iff } a=b=c\ (\text{QED})$$

1756. If $a,b,c>0$, then prove that :

$$(a^2+b^2+c^2)(b+c-a)(c+a-b)(a+b-c)\leq abc(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$= s, R, r\ (say);\ and\ then: \sum_{cyc} a = s \to (1), a = s - x, b = s - y, c = s - z \to (2),$$

$$abc = r^2 s \to (3), \sum_{cyc} ab = 4Rr + r^2 \to (4), \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (5)\ and\ so,$$

$$(a^2 + b^2 + c^2)(a + b - c)(b + c - a)(c + a - b) \overset{via\ (1),(2)\ and\ (5)}{=}$$

$$(s^2 - 8Rr - 2r^2)(x - (s - x))(y - (s - y))(z - (s - z))$$

$$= (s^2 - 8Rr - 2r^2)(2x - s)(2y - s)(2z - s)$$

$$= (s^2 - 8Rr - 2r^2)\left(-s^3 + 2s^2 \sum_{cyc} x - 4s \sum_{cyc} xy + 8xyz\right)$$

$$= (s^2 - 8Rr - 2r^2)(-s^3 + 2s^2(2s) - 4s(s^2 + 4Rr + r^2) + 32Rrs)$$

$$= -s(s^2 - 8Rr - 2r^2)(s^2 - 16Rr + 4r^2) \therefore RHS - LHS \overset{via\ (3)\ and\ (4)}{=}$$

$$r^2 s(4Rr + r^2) + s(s^2 - 8Rr - 2r^2)(s^2 - 16Rr + 4r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow s^4 - (24Rr - 2r^2)s^2 + r^2(128R^2 + 4Rr - 7r^2) \boxed{\underset{(*)}{\overset{?}{\geq}}} 0\ and$$

$$\because (s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove}:$$

$$\text{LHS of } (*) \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (2R - 2r)s^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} r(32R^2 - 41Rr + 8r^2)$$

$$\text{Now, } (2R - 2r)s^2 \overset{Rouche}{\geq} (2R - 2r)\left(2R^2 + 10Rr - r^2 - 2(R - 2r).\sqrt{R^2 - 2Rr}\right)$$

$$\overset{?}{\geq} r(32R^2 - 41Rr + 8r^2)$$

$$\Leftrightarrow (R - 2r)(4R^2 - 8Rr + 3r^2) \boxed{\underset{(***)}{\overset{?}{\geq}}} 2(R - 2r)(2R - 2r).\sqrt{R^2 - 2Rr}\ and \because$$

$$R - 2r \overset{Euler}{\geq} 0 \therefore \text{in order to prove } (***), \text{it suffices to prove}:$$

$$(4R^2 - 8Rr + 3r^2)^2 \overset{?}{>} 4(R^2 - 2Rr)(2R - 2r)^2 \Leftrightarrow r^2(8R(R - 2r) + 9r^2) \overset{?}{>} 0$$

$$\to \text{true} \therefore (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (a^2 + b^2 + c^2)(b + c - a)(c + a - b)(a + b - c) \leq abc(ab + bc + ca)$$

$$\forall\ a, b, c > 0, '' ='' \text{ iff } a = b = c\ (QED)$$

1757. **If $a, b, c > 0$, then prove that:**

$$(a + b + c)^3(a + b - c)(b + c - a)(c + a - b) \leq 27a^2b^2c^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0,$$

$$y+z-x=2a>0 \text{ and } z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); and then : $\sum_{cyc} a = s \to (1), a = s-x, b = s-y, c = s-z \to (2),$

$$abc = r^2 s \to (3) \text{ and}, (a+b+c)^3(a+b-c)(b+c-a)(c+a-b) \overset{\text{via (1) and (2)}}{=}$$
$$s^3\big(x-(s-x)\big)\big(y-(s-y)\big)\big(z-(s-z)\big) = s^3(2x-s)(2y-s)(2z-s)$$
$$= s^3\left(-s^3+2s^2\sum_{cyc} x - 4s\sum_{cyc} xy + 8xyz\right)$$
$$= s^3(-s^3+2s^2(2s)-4s(s^2+4Rr+r^2)+32Rrs) = -s^4(s^2-16Rr+4r^2)$$
$$\therefore \text{RHS} - \text{LHS} \overset{\text{via (3)}}{=} 27r^4s^2 + s^4(s^2-16Rr+4r^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow s^4-(16Rr-4r^2)s^2+27r^4 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0,$$
$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove : LHS of } (*) \overset{?}{\geq} (s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (8R-3r)s^2 \underset{(**)}{\overset{?}{\geq}} r(128R^2-80Rr-r^2)$$
$$\text{Now}, (8R-3r)s^2 \overset{\text{Rouche}}{\geq} (8R-3r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$
$$\overset{?}{\geq} r(128R^2-80Rr-r^2)$$
$$\Leftrightarrow 2(R-2r)(8R^2-11Rr-r^2) \underset{(***)}{\overset{?}{\geq}} 2(R-2r)(8R-3r).\sqrt{R^2-2Rr} \text{ and } \because$$
$$R-2r \overset{\text{Euler}}{\geq} 0 \therefore \text{ in order to prove } (***), \text{ it suffices to prove :}$$
$$(8R^2-11Rr-r^2)^2 \overset{?}{>} (R^2-2Rr)(8R-3r)^2 \Leftrightarrow r^3(40R+r) \overset{?}{>} 0 \to \text{true}$$
$$\therefore (***) \Rightarrow (**) \Rightarrow (*) \text{is true} \therefore (a+b+c)^3(a+b-c)(b+c-a)(c+a-b)$$
$$\leq 27a^2b^2c^2\ \forall a,b,c>0, ''='' \text{ iff } a=b=c \text{ (QED)}$$

1758. If $a,b,c>0$, $a+b+c\leq 3$ then:

$$\frac{a}{b^{2025}}+\frac{b}{c^{2025}}+\frac{c}{a^{2025}} \geq a+b+c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a}{b^{2025}}+\frac{b}{c^{2025}}+\frac{c}{a^{2025}} = \sum \frac{a}{b^{2025}} = \sum \frac{a^{2026}}{(ab)^{2025}} \overset{Radon}{\geq}$$

$$\geq \frac{(a+b+c)^{2026}}{(ab+bc+ca)^{2025}} \geq \frac{(a+b+c)^{2026}}{\left(\frac{(a+b+c)^2}{3}\right)^{2025}} = \frac{(a+b+c).(a+b+c)^{2025}.3^{2025}}{(a+b+c)^{2025}.(a+b+c)^{2025}} =$$

$$= (a+b+c).\frac{3^{2025}}{(a+b+c)^{2025}} \overset{a+b+c\leq 3}{\geq} (a+b+c).\frac{3^{2025}}{3^{2025}} = (a+b+c)$$

Equality holds for a=b=c=1.

1759. If $a,b>0\,,a+b=1$ then:

$$\frac{1}{a^2+ab+b^2}+\frac{4a^2b^2+1}{ab} \geq \frac{19}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ m = ab \overset{AM-GM}{\leq} \frac{(a+b)^2}{4} \overset{a+b=1}{=} \frac{1}{4}\ (1)$$

$$a^2+ab+b^2 = (a+b)^2-2ab+ab = (1-m)\ (2)$$

$$We\ need\ to\ show:$$

$$\frac{1}{a^2+ab+b^2}+\frac{4a^2b^2+1}{ab} \geq \frac{19}{3}\ or$$

$$\frac{1}{1-m}+\frac{4m^2+1}{m} \geq \frac{19}{3}\ or\ \frac{1+4m^2-4m^3}{m(1-m)} \geq \frac{19}{3}$$

$$12m^3-31m^2+19m-3 \leq 0\ or (4m-1)\{(m-2)(3m-1)+1\} \leq 0$$

$$true\ since\ m \leq \frac{1}{4}\ then\ (m-2)<0\ and\ (3m-1) = 3\left(m-\frac{1}{3}\right)<0$$

$$(4m-1)<0\ \ then(4m-1)\{(m-2\ (m-2)(3m-1)+1\}<0$$

Equality holds for $a=b=\frac{1}{2}$.

1760. If $a,b,c>0$ then:

$$\frac{a}{3a^2+2b^2+c^2}+\frac{a}{3a^2+2b^2+c^2}+\frac{a}{3a^2+2b^2+c^2}\le\frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3a^2+2b^2+c^2=2(a^2+b^2)+(a^2+c^2)\overset{AM-GM}{\ge}4ab+2ac=2a(2b+c)\ (1)$$

$$\frac{a}{3a^2+2b^2+c^2}+\frac{a}{3a^2+2b^2+c^2}+\frac{a}{3a^2+2b^2+c^2}=$$

$$=\sum\frac{a}{3a^2+2b^2+c^2}\overset{(1)}{\le}\frac{1}{2}\sum\frac{1}{2b+c}=$$

$$=\frac{1}{2}\sum\frac{1}{b+b+c}\overset{AM-HM}{\le}\frac{1}{2}\cdot\frac{1}{9}\sum\left(\frac{1}{b}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{18}\cdot 3\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=\frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Equality holds for a =b=c.

1761. If $a,b,c>0,\ ab+bc+ca=abc$ then:

$$\frac{a^2}{a+bc}+\frac{b^2}{b+ac}+\frac{c^2}{c+ab}\ge\frac{a+b+c}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^2+abc\overset{abc=ab+bc+ca}{=}a^2+ab+bc+ca=$$

$$=(a+c)(a+b)\overset{AM-GM}{\le}\frac{\big((a+b)+(a+c)\big)^2}{4}=\frac{(2a+b+c)^2}{4}\ (1)$$

$$\frac{a^2}{a+bc}+\frac{b^2}{b+ac}+\frac{c^2}{c+ab}=\sum\frac{a^2}{a+bc}=$$

$$=\sum\frac{a^3}{a^2+abc}\overset{(1)}{\ge}\sum\frac{4a^3}{(2a+b+c)^2}\overset{Radon}{\ge}\frac{4(a+b+c)^3}{\big(4(a+b+c)\big)^2}=\frac{a+b+c}{4}$$

Equality holds for a=b=c=3.

1762. If $a,b>0$ then:

$$\frac{ab}{a^2+b^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2(a^2+b^2)}\ge\frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ \frac{a^2+b^2}{ab}=t\overset{AM-GM}{\ge}\frac{2ab}{ab}=2\ (1)$$

$$We\ need\ to\ show:$$

$$\frac{ab}{a^2+b^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{2(a^2+b^2)}\ge\frac{9}{2}$$

$$or\ \frac{ab}{a^2+b^2}+\frac{a+b}{ab}\sqrt{2.\frac{(a+b)^2}{2}}\overset{CBS}{\ge}\frac{9}{2}$$

$$or\ \frac{ab}{a^2+b^2}+\frac{(a+b)^2}{ab}\ge\frac{9}{2}\ or\ \frac{ab}{a^2+b^2}+\frac{a^2+b^2}{ab}+2\ge\frac{9}{2}$$

$$or\frac{1}{t}+t\overset{\frac{a^2+b^2}{ab}=t}{\ge}\frac{5}{2}\ or\ 2t^2-5t+2\ge 0\ or\ (t-2)(2t-1)\ge 0\ \ true\ \ by\ \ (1)$$

Equality holds for a=b=1

1763. If $x,y,z>0\ ,x+y+z=3$ then:

$$\sum\frac{x^3}{x^2+y^2}\ge\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{x^3}{x^2+y^2}=\sum\left(x-\frac{xy^2}{x^2+y^2}\right)\overset{AM-GM}{\ge}\sum\left(x-\frac{xy^2}{2xy}\right)=$$

$$=\sum\left(x-\frac{y}{2}\right)=\frac{\sum x}{2}=\frac{3}{2}$$

$$Equality\ holds\ for\ x=y=z=1.$$

1764. If $a, b, c > 0$ and $a^2 + b^2 = c^2$ then:

$$\frac{(a+b)(b+c)(c+a)}{abc} \geq 4 + 3\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ a = c\cos\theta, b = c\sin\theta$$

$$Let\ t = \cos\theta + \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) =$$

$$= \sqrt{2}\sin\left(\frac{\pi}{4} + \theta\right) \leq \sqrt{2}\ \left(as\ \sin\left(\frac{\pi}{4} + \theta\right) \leq 1\right)$$

$$\sin\theta . \cos\theta = \frac{(\cos\theta + \sin\theta)^2 - 1}{2} = \frac{t^2 - 1}{2}$$

Putting the value of above mentioned a, c we get:

$$\frac{(a+b)(b+c)(c+a)}{abc} = \frac{(\cos\theta + \sin\theta)(1 + \sin\theta)(1 + \cos\theta)}{\sin\theta . \cos\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(1 + \sin\theta + \cos\theta + \sin\theta\cos\theta)}{\sin\theta . \cos\theta} = \frac{t\left(1 + t + \frac{t^2 - 1}{2}\right)}{\frac{t^2 - 1}{2}} =$$

$$= \frac{t(t+1)^2}{t^2 - 1} = \frac{t(t+1)}{t-1}$$

$$(putting\ the\ value\ of\ (\cos\theta + \sin\theta) \&\ \cos\theta . \sin\theta)$$

We need to show:

$$\frac{t(t+1)}{t-1} \geq 4 + 3\sqrt{2}\ or\ t^2 - t(3 + 3\sqrt{2}) + (4 + 3\sqrt{2}) \geq 0$$

$$or\ (t - \sqrt{2})\left(t - (3 + 2\sqrt{2})\right) \geq 0\ true\ (as\ t \leq \sqrt{2})$$

Equality holds for a=b =x and c=$\sqrt{2}x$ *where* $x > 0$.

1765. If $a, b, c > 0$ then:

$$\left(\frac{a}{b+c} + \frac{b}{c+a}\right)\left(\frac{b}{c+a} + \frac{c}{a+b}\right)\left(\frac{c}{a+b} + \frac{a}{b+c}\right) \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ a+b\ge a+c\ge b+c\ and\ \frac{1}{b+c}\ge\frac{1}{c+a}\ge\frac{1}{a+b}$$

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\overset{Chebyshev}{\ge}\frac{1}{2}(a+b)\left(\frac{1}{b+c}+\frac{1}{c+a}\right)\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2}(a+b)\cdot 2\sqrt{\frac{1}{(b+c)(c+a)}}=\frac{a+b}{\sqrt{(b+c)(c+a)}}$$

Similarly:

$$\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\ge\frac{b+c}{\sqrt{(b+a)(c+a)}},$$

$$\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\ge\frac{c+a}{\sqrt{(b+a)(c+b)}}$$

Using above result we get :

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\ge$$

$$\ge\frac{a+b}{\sqrt{(b+c)(c+a)}}\cdot\frac{b+c}{\sqrt{(b+a)(c+a)}}\cdot\frac{c+a}{\sqrt{(b+a)(c+b)}}=1$$

Equality holds for a=b=c=1.

1766. If $a,b,c>0,\ a^2+b^2+c^2=3$ then:

$$\frac{a^3+b^3}{a+2b}+\frac{b^3+c^3}{b+2c}+\frac{c^3+a^3}{c+2a}\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a^3}{a+2b}=\sum\left(a^2-\frac{2a^2b}{a+2b}\right)=$$

$$=\sum a^2-2\sum\frac{a^2b}{a+b+b}\overset{AM-HM}{\ge}\sum a^2-\frac{2}{9}\sum(ab+a^2+a^2)=$$

$$=\sum a^2-\frac{2}{9}\sum ab-\frac{4}{9}\sum a^2\ge\sum a^2-\frac{2}{9}\sum a^2-\frac{4}{9}\sum a^2=\frac{1}{3}\sum a^2=\frac{1}{3}.3=1$$

$$\sum\frac{b^3}{a+2b}=\frac{1}{2}\sum\left(b^2-\frac{ab^2}{a+b+b}\right)=\frac{1}{2}\sum b^2-\frac{1}{2}\sum\frac{ab^2}{a+b+b}\overset{AM-HM}{\geq}$$

$$\geq\frac{1}{2}\sum b^2-\frac{1}{2}\cdot\frac{1}{9}\left(\sum b^2+\sum ab+\sum ab\right)\geq$$

$$\geq\frac{1}{2}\sum b^2-\frac{1}{18}\left(\sum b^2+\sum b^2+\sum b^2\right)=\frac{1}{3}\sum b^2=\frac{1}{3}.3=1$$

$$\frac{a^3+b^3}{a+2b}+\frac{b^3+c^3}{b+2c}+\frac{c^3+a^3}{c+2a}=\sum\frac{a^3+b^3}{a+2b}=\sum\frac{a^3}{a+2b}+\sum\frac{b^3}{a+2b}\geq 1+1=2$$

Equality holds for $a=b=c=1$.

1767. If $x,y,z>0$ then:

$$\sqrt{\frac{x}{z+3x}}+\sqrt{\frac{y}{x+3y}}+\sqrt{\frac{z}{y+3z}}\leq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{x}{z+3x}=\frac{1}{3}\sum\left(1-\frac{z}{z+3x}\right)=\frac{1}{3}\left(3-\sum\frac{z}{z+3x}\right)=$$

$$=\frac{1}{3}\left(3-\sum\frac{z^2}{z^2+3xz}\right)\overset{CBS}{\leq}\frac{1}{3}\left(3-\frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}\right)=$$

$$=\frac{1}{3}\left(2+1-\frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}\right)=$$

$$=\frac{1}{3}\left(2+\frac{xy+yz+zx}{x^2+y^2+z^2+3(xy+yz+zx)}\right)=$$

$$\leq\frac{1}{3}\left(2+\frac{xy+yz+zx}{xy+yz+zx+3(xy+yz+zx)}\right)=\frac{1}{3}\left(2+\frac{1}{4}\right)=\frac{1}{3}\cdot\frac{9}{4}\ (1)$$

$$\sqrt{\frac{x}{z+3x}}+\sqrt{\frac{y}{x+3y}}+\sqrt{\frac{z}{y+3z}}\overset{CBS}{\leq}\sqrt{3\sum\frac{x}{z+3x}}\overset{(1)}{\leq}\sqrt{3,\frac{1}{3}\cdot\frac{9}{4}}=\frac{3}{2}$$

Equality holds for $x=y=z$.

1768. If $x,y,z>0,\ x+y+z=6$ then:

$$\frac{xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\le 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Lemma:\forall\ a,b,c\in R^+\ such\ that\ a+b+c=3\ then\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ge ab+bc+ca$$

$$Proof:3(a+b+c)=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$Hence:\sum ab=\frac{1}{2}(3a-a^2+3b-b^2+3c-c^2)$$

$$then\ \sqrt{a}+\sqrt{b}+\sqrt{c}-(ab+bc+ca)$$

$$=\sqrt{a}+\sqrt{b}+\sqrt{c}-\frac{1}{2}(3a-a^2+3b-b^2+3c-c^2)=\frac{1}{2}\sum(a^2-3a+2\sqrt{a})=$$

$$=\frac{1}{2}\sum\left(\sqrt{a}(\sqrt{a}-1)^2(\sqrt{a}+2)\right)\ge 0\ true$$

$$Back\ to\ the\ main\ problem\ \ x+y+z=6\ or\ \frac{x}{2}+\frac{y}{2}+\frac{z}{2}=3$$

$$\sqrt{\frac{x}{2}}+\sqrt{\frac{y}{2}}+\sqrt{\frac{z}{2}}\overset{Lemma}{\ge}\frac{xy}{4}+\frac{yz}{4}+\frac{zx}{4}\ or$$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\ge\frac{xy+yz+zx}{2\sqrt{2}}\ or\ \frac{xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\le 2\sqrt{2}$$

Equality holds for $x=y=z=2$.

1769. If $a,b,c>0$ then:

$$4a^2+5b^2+c^2+\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\ge 4b(a+c)+\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{a+c}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$4a^2+5b^2+c^2=(4a^2+b^2)+(4b^2+c^2)\overset{AM-GM}{\ge}4ab+4bc=4b(a+c)\ (1)$$

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\sum\frac{a^2}{a+b}=\sum\frac{a^2-b^2+b^2}{a+b}=\sum\frac{(a+b)(a-b)+b^2}{a+b}=$$

$$=\sum(a-b)+\sum\frac{b^2}{a+b}=\sum\frac{b^2}{a+b}=\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{a+c}\quad(2)$$

$By\ (1)\&\ (2)\ we\ get:$

$$4a^2+5b^2+c^2+\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq 4b(a+c)+\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{a+c}$$

Equality holds for $a=1,b=2,c=4$.

1770. If $a,b,c>0$ and $a^3+b^3+c^3=81$ then:

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ac}+\frac{1}{c^2+2ab}\geq\frac{1}{9}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

$$81=a^3+b^3+c^3=\frac{a^3}{1^2}+\frac{b^3}{1^2}+\frac{c^3}{1^2}\overset{RADON}{\geq}\frac{(a+b+c)^3}{9}$$

$$(a+b+c)^3\leq 81\times 9$$

$$a+b+c\leq 9$$

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ac}+\frac{1}{c^2+2ab}=\sum\frac{1}{a^2+2bc}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{(1+1+1)^2}{a^2+b^2+c^2+2(ab+bc+ca)}=\frac{9}{(a+b+c)^2}\overset{a+b+c\leq 9}{\geq}\frac{9}{9^2}=\frac{1}{9}$$

Equality holds for $a=b=c=3$.

1771. If $a,b,c>0,\ a^2+b^2+c^2=3,\lambda\geq 0$ then:

$$\sum\frac{a^3}{a+\lambda b}\geq\sum\frac{a^2}{\lambda+a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a^2}{\lambda+a^2}=\sum\left(1-\frac{\lambda}{\lambda+a^2}\right)=3-\lambda\sum\frac{1}{\lambda+a^2}\overset{CBS}{\leq}$$

$$\leq 3-\frac{(1+1+1)^2}{3\lambda+a^2+b^2+c^2}\overset{a^2+b^2+c^2=3}{=}3-\frac{9\lambda}{3\lambda+3}=\frac{3}{\lambda+1}\ (A)$$

$$\sum\frac{a^3}{a+\lambda b}=\sum\frac{a^4}{a^2+\lambda ab}\overset{CBS}{\geq}\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\lambda(ab+bc+ca)}\geq$$

$$\geq\frac{9}{3+\lambda(a^2+b^2+c^2)}\overset{a^2+b^2+c^2=3}{=}\frac{9}{3+3\lambda}=\frac{3}{\lambda+1}\ (B)$$

$By\ (A)\ \&\ (B) we\ have:$

$$\sum\frac{a^3}{a+\lambda b}\geq\sum\frac{a^2}{\lambda+a^2}$$

Equality holds for $a=b=c=1$.

1772. **If $a,b,c>0, abc=1, \lambda\geq 0$ then:**

$$\sum\frac{bc}{a^2(b+\lambda c)}\geq\frac{ab+bc+ca}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{bc}{a^2(b+\lambda c)}=\sum\frac{abc}{a^3(b+\lambda c)}\overset{abc=1}{=}\sum\frac{\left(\frac{1}{a}\right)^3}{(b+\lambda c)}\overset{Holder}{\geq}$$

$$\geq\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3}{3(\lambda+1)(a+b+c)}=\frac{\left(\frac{ab+bc+ca}{abc}\right)^3}{3(\lambda+1)(a+b+c)}\overset{abc=1}{=}$$

$$=\frac{(ab+bc+ca)^3}{3(\lambda+1)(a+b+c)}=\frac{ab+bc+ca}{3(\lambda+1)}\cdot\frac{(ab+bc+ca)^2}{(a+b+c)}\geq$$

$$\geq\frac{ab+bc+ca}{3(\lambda+1)}\cdot\frac{3(ab.bc+bc.ca+ca.ab)}{(a+b+c)}=$$

$$=\frac{ab+bc+ca}{3(\lambda+1)}\cdot\frac{3abc(a+b+c)}{(a+b+c)}\overset{abc=1}{=}\frac{ab+bc+ca}{\lambda+1}$$

Equality holds for $a=b=c=1$.

1773. If $x, y > 0, xy = 1$ and $n \in \mathbb{N}, 0 \le \lambda \le 1$, then :

$$\frac{x^{n+1}}{y^n + \lambda} + \frac{y^{n+1}}{x^n + \lambda} \ge \frac{2}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{WLOG we may assume : } x \ge y \text{ and then : } \frac{x^n}{y^n + \lambda} \ge \frac{y^n}{x^n + \lambda} \text{ and so,}$$

$$\text{LHS} \overset{\text{Chebyshev}}{\ge} \frac{x+y}{2} \cdot \left(\frac{x^n}{y^n + \lambda} + \frac{y^n}{x^n + \lambda}\right) \overset{\text{AM-GM}}{\ge} \sqrt{xy} \cdot \left(\frac{a^2 + b^2 + (a+b)\lambda}{ab + (a+b)\lambda + \lambda^2}\right)$$

$$(a = x^n, b = y^n) \overset{xy=1}{=} \frac{a^2 + b^2 + (a+b)\lambda}{ab + (a+b)\lambda + \lambda^2} \overset{?}{\ge} \frac{2}{\lambda + 1}$$

$$\Leftrightarrow a^2 + b^2 + (a+b)\lambda + \lambda(a^2 + b^2) + (a+b)\lambda^2 \overset{?}{\ge} 2ab + 2(a+b)\lambda + 2\lambda^2$$

$$\Leftrightarrow (a-b)^2 + \lambda\left(a^2 + b^2 - (a+b)\right) + \lambda^2(a+b-2) \overset{?}{\ge} 0 \to \text{true} \because a^2 + b^2 \ge$$

$$\frac{1}{2}(a+b)^2 \overset{\text{AM-GM}}{\ge} \frac{1}{2}(a+b).2\sqrt{ab} \overset{ab=1}{=} a+b \Rightarrow a^2 + b^2 - (a+b) \ge 0$$

$$\text{and} \because a+b \overset{\text{AM-GM}}{\ge} 2\sqrt{ab} \overset{ab=1}{=} 2 \Rightarrow a+b-2 \ge 0 \text{ and} \because \lambda \ge 0 \therefore \frac{x^{n+1}}{y^n + \lambda} + \frac{y^{n+1}}{x^n + \lambda}$$

$$\ge \frac{2}{\lambda + 1} \; \forall x, y > 0 \,\middle|\, xy = 1 \text{ and } n \in \mathbb{N}, 0 \le \lambda \le 1, \text{"="} \text{ iff } x = y = 1 \text{ (QED)}$$

1774. If $x, y > 0, xy = 1$ and $n \in \mathbb{N}$, then :

$$\frac{x^{n+1}}{y^n + 1} + \frac{y^{n+1}}{x^n + 1} \ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall first prove that : } \frac{x^{n+1}}{y^n + \lambda} + \frac{y^{n+1}}{x^n + \lambda} \ge \frac{2}{\lambda + 1} \; \forall \lambda \ge 0$$

$$\text{WLOG we may assume : } x \ge y \text{ and then : } \frac{x^n}{y^n + \lambda} \ge \frac{y^n}{x^n + \lambda} \text{ and so,}$$

$$\text{LHS} \overset{\text{Chebyshev}}{\ge} \frac{x+y}{2} \cdot \left(\frac{x^n}{y^n + \lambda} + \frac{y^n}{x^n + \lambda}\right) \overset{\text{AM-GM}}{\ge} \sqrt{xy} \cdot \left(\frac{a^2 + b^2 + (a+b)\lambda}{ab + (a+b)\lambda + \lambda^2}\right)$$

$$(a = x^n, b = y^n) \overset{xy=1}{=} \frac{a^2 + b^2 + (a+b)\lambda}{ab + (a+b)\lambda + \lambda^2} \overset{?}{\ge} \frac{2}{\lambda + 1}$$

$$\Leftrightarrow a^2+b^2+(a+b)\lambda+\lambda(a^2+b^2)+(a+b)\lambda^2 \overset{?}{\geq} 2ab+2(a+b)\lambda+2\lambda^2$$

$$\Leftrightarrow (a-b)^2+\lambda\left(a^2+b^2-(a+b)\right)+\lambda^2(a+b-2) \overset{?}{\geq} 0 \to \text{true} \because a^2+b^2 \geq$$

$$\frac{1}{2}(a+b)^2 \overset{AM-GM}{\geq} \frac{1}{2}(a+b).2\sqrt{ab} \overset{ab=1}{=} a+b \Rightarrow a^2+b^2-(a+b) \geq 0$$

$$\text{and} \because a+b \overset{AM-GM}{\geq} 2\sqrt{ab} \overset{ab=1}{=} 2 \Rightarrow a+b-2 \geq 0 \text{ and} \because \lambda \geq 0 \therefore \frac{x^{n+1}}{y^n+\lambda}+\frac{y^{n+1}}{x^n+\lambda}$$

$$\overset{(*)}{\geq} \frac{2}{\lambda+1} \ \forall x,y>0 \,|\, xy=1 \text{ and } n \in \mathbb{N}, \lambda \geq 0 \text{ and putting } \lambda = 1 \text{ in } (*), \text{we get}:$$

$$\frac{x^{n+1}}{y^n+1}+\frac{y^{n+1}}{x^n+1} \geq 1 \ \forall x,y>0 \,|\, xy=1 \text{ and } n \in \mathbb{N}, ''='' \text{ iff } x=y=1 \text{ (QED)}$$

1775. If $a,b,c>0$ and $\lambda \geq 0$ then :

$$\sum_{cyc} \frac{a}{b+\lambda c} \geq \frac{4}{\lambda+1} - \frac{\sum_{cyc} ab}{(\lambda+1)\sum_{cyc} a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{a}{b+\lambda c} = \sum_{cyc} \frac{a^2}{ab+\lambda ca} \overset{Bergstrom}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{(\lambda+1)\sum_{cyc} ab}$$

$$\Rightarrow \sum_{cyc} \frac{a}{b+\lambda c} + \frac{\sum_{cyc} ab}{(\lambda+1)\sum_{cyc} a^2} \geq \frac{\left(\sum_{cyc} a\right)^2}{(\lambda+1)\sum_{cyc} ab} + \frac{\sum_{cyc} ab}{(\lambda+1)\sum_{cyc} a^2}$$

$$= \frac{1}{\lambda+1}\cdot\left(\frac{m+2n}{n}+\frac{n}{m}\right)\left(m=\sum_{cyc} a^2, n=\sum_{cyc} ab\right) = \frac{1}{\lambda+1}\cdot\left(\frac{(m+n)^2}{mn}\right)$$

$$\overset{AM-GM}{\geq} \frac{4}{\lambda+1} \Rightarrow \sum_{cyc} \frac{a}{b+\lambda c} \geq \frac{4}{\lambda+1} - \frac{\sum_{cyc} ab}{(\lambda+1)\sum_{cyc} a^2} \ \forall a,b,c>0, \lambda \geq 0,$$

$$''='' \text{ iff } a=b=c \text{ (QED)}$$

1776. Let $a,b,c>0$. Prove that :

$$\left(\sum_{cyc} \sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc} \sqrt{\frac{a^2+ab+b^2}{3}}\right) + \frac{9(a^3+b^3+c^3)}{(a+b+c)(ab+bc+ca)} \geq 12$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Morocco

By AM − GM inequality, we have

$$(a+b)\sqrt[3]{\frac{a^3+b^3}{2}}=2\sqrt[3]{\left(\frac{(a+b)^2}{4}\right)^2.(a^2-ab+b^2)}$$
$$\le\frac{2}{3}\left(2.\frac{(a+b)^2}{4}+(a^2-ab+b^2)\right)=a^2+b^2$$

$$\Rightarrow\sqrt[3]{\frac{2}{a^3+b^3}}\ge\frac{a+b}{a^2+b^2}.\text{Also, we have }\sqrt{\frac{a^2+ab+b^2}{3}}=\sqrt{\frac{(a+b)^2}{4}+\frac{(a-b)^2}{12}}$$
$$\ge\frac{a+b}{2}.$$

If $a\ge b\ge c$, **we have** $\frac{a+b}{2}\ge\frac{c+a}{2}\ge\frac{b+c}{2}$ **and** $\sqrt[3]{\frac{2}{a^3+b^3}}\le\sqrt[3]{\frac{2}{c^3+a^3}}$
$\le\sqrt[3]{\frac{2}{b^3+c^3}}$, **then by**

Chebyshev and CBS inequalities, we have

$$\left(\sum_{cyc}\sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc}\sqrt{\frac{a^2+ab+b^2}{3}}\right)\ge\left(\sum_{cyc}\sqrt[3]{\frac{2}{a^3+b^3}}\right)\left(\sum_{cyc}\frac{a+b}{2}\right)$$
$$\ge3\sum_{cyc}\frac{a+b}{2}.\sqrt[3]{\frac{2}{a^3+b^3}}$$
$$\ge\frac{3}{2}\sum_{cyc}\frac{(a+b)^2}{a^2+b^2}\ge\frac{3(a+b+c)^2}{a^2+b^2+c^2}=3+\frac{6(ab+bc+ca)}{a^2+b^2+c^2}.\quad(1)$$

Now, by Schur's inequality, we have

$$\frac{3(a^3+b^3+c^3)}{(a+b+c)(ab+bc+ca)}$$
$$\ge\frac{2(a^3+b^3+c^3)+a^2(b+c)+b^2(c+a)+c^2(a+b)-3abc}{(a+b+c)(ab+bc+ca)}$$

$$=\frac{(a+b+c)(2a^2+2b^2+2c^2-ab-bc-ca)}{(a+b+c)(ab+bc+ca)}=\frac{2(a^2+b^2+c^2)}{ab+bc+ca}-1.\ \ (2)$$

From (1) and (2), it suffices to prove that

$$\frac{6(ab+bc+ca)}{a^2+b^2+c^2}+\frac{6(a^2+b^2+c^2)}{ab+bc+ca}\ge 12,$$

which is true by AM − GM inequlity.

So the proof is complete. Equality holds iff $a=b=c$.

1777. Let $a,b,c\ge 0, a+b+c=4$. Prove that :

$$\sqrt{9b-bc+9c}+\sqrt{9c-ca+9a}+\sqrt{9a-ab+9b}\le 10\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c=4,\ \ q:=ab+bc+ca\le\frac{p^2}{3}=\frac{16}{3},\ \ r:=abc.$

By CBS inequality, we have

$$\sum_{cyc}\sqrt{9b-bc+9c}\le\sqrt{\sum_{cyc}(9b-bc+9c)(a+6)\cdot\sum_{cyc}\frac{1}{a+6}}=$$

$$=\sqrt{(108p+12q-3r)\cdot\frac{12p+q+108}{216+36p+6q+r}}\overset{?}{\le}10\sqrt{2}$$

$$\overset{p=4}{\Leftrightarrow}\ 4608-1104q-12q^2+(668+3q)r\ge 0.\ (*)$$

If $q\le 4:\ LHS_{(*)}\ge 4608-1104q-12q^2=(4-q)(1152+12q)\ge 0.$

If $4\le\ q\le\frac{16}{3}:\ LHS_{(*)}\overset{Schur}{\ge}\ 4608-1104q-12q^2+\frac{(668+3q)(4pq-p^3)}{9}=$

$$=\frac{20(q-4)(16-3q)}{9}\ge 0.$$

Equality holds iff $a=b=c=\frac{4}{3}$ & $a=b=2, c=0$ and its permutation.

1778. If $a, b, c>0$ and $abc=1$, then prove that :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\sum_{cyc}\frac{b+c}{a}+2\sum_{cyc}\frac{b}{a}\geq$$

$$\frac{\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc}{abc}+\frac{2\sum_{cyc}a}{\sqrt[3]{abc}}\overset{abc=1}{=}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3+2\sum_{cyc}a\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{abc}.\left(\sum_{cyc}ab\right)-3+2\sum_{cyc}a\overset{abc=1}{=}$$

$$=2\sum_{cyc}ab+\sum_{cyc}ab-3+2\sum_{cyc}a\overset{abc=1}{=}$$

$$=\frac{2\sum_{cyc}ab}{abc}+2\sum_{cyc}a+\sum_{cyc}ab-3=$$

$$=2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\sum_{cyc}ab-3\overset{AM-GM}{\geq}2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$+3\sqrt[3]{a^2b^2c^2}-3\overset{abc=1}{=}2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\therefore\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\forall\, a,b,c>0\,\middle|\,abc=1,''=''\text{ iff } a=b=c=1\text{ (QED)}$$

1779. Let $a, b, c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\sqrt{a+ab+b}+\sqrt{b+bc+c}+\sqrt{c+ca+a}\leq\sqrt{5a+5b+5c+12}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sum_{cyc}\sqrt{b+bc+c} \le \sqrt{\sum_{cyc}(b+bc+c)(a+2)\,.\sum_{cyc}\frac{1}{a+2}}$$

From the given condition, we have

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}=1.$$

So it suffices to prove that

$$\sum_{cyc}(b+bc+c)(a+2) \le 5a+5b+5c+12 \text{ or } ab+bc+ca \le a+b+c.$$

Let $p := a+b+c, q := ab+bc+ca, r := abc$. We have $4 = q + r \le \frac{p^2}{3}+\frac{p^3}{27}$, then $p \ge 3$.

If $p \ge 4:\ ab+bc+ca \le 4 \le a+b+c.$

If $3 \le p \le 4:\ 4pq \overset{Schur}{\le} p^3+9r = p^3+9(4-q) \Rightarrow q \le \frac{p^3+36}{4p+9}$

$$= p - \frac{(p^2-9)(4-p)}{4p+9} \le p.$$

Equality holds if $a=b=c=1$ & $a=b=2, c=0$ and permutations.

1780. If $a,b,c>0$ then:

$$\frac{a^3+2a^2b}{a^2+3ab+b^2}+\frac{b^3+2b^2c}{b^2+3bc+c^2}+\frac{c^3+2c^2a}{c^2+3ca+a^2} \ge \frac{3(a+b+c)}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a^3+2a^2b}{a^2+3ab+b^2}+\frac{b^3+2b^2c}{b^2+3bc+c^2}+\frac{c^3+2c^2a}{c^2+3ca+a^2}=$$

$$=\sum\frac{a^3+2a^2b}{a^2+3ab+b^2}=\sum\frac{a^2(a+2b)}{a(a+2b)+b(a+b)}=$$

$$=\sum\left(a-\frac{ab(a+b)}{a(a+2b)+b(a+b)}\right)=$$

$$=\sum a-\sum\frac{ab(a+b)}{a^2+3ab+b^2}\overset{AM-GM}{\geq}\sum a-\sum\frac{ab(a+b)}{2ab+3ab}=$$

$$=\sum a-\frac{\sum(a+b)}{5}=\frac{3(a+b+c)}{5}$$

Equality holds for $a=b=c$.

1781. Let $a,b,c\geq 0, a+b+c=ab+bc+ca>0$. Prove that :

$$\frac{a(3a+5bc)}{b+c}+\frac{b(3b+5ca)}{c+a}+\frac{c(3c+5ab)}{a+b}\geq 12$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c,\ \ q:=ab+bc+ca,\ \ r:=abc$. We have $p=q\leq\frac{p^2}{3}$, then $p\geq 3$.

By AM – GM inequality, we have

$$\frac{a^2}{b+c}+\frac{a^2(b+c)}{4}\geq a^2\ \ (\text{and analogs})$$

$$\Rightarrow\sum_{cyc}\frac{a^2}{b+c}\geq a^2+b^2+c^2-\frac{1}{4}\sum_{cyc}a^2(b+c)$$

$$=p^2-2q-\frac{1}{4}(pq-3r)\overset{q=p}{\triangleq}\frac{3p^2}{4}-2p+\frac{3r}{4}.\ \ (1)$$

Then

$$\sum_{cyc}\frac{a(3a+5bc)}{b+c}\overset{(1)\,\&\,CBS}{\geq}3\left(\frac{3p^2}{4}-2p+\frac{3r}{4}\right)+5abc.\frac{9}{2(a+b+c)}$$

$$=\frac{9p^2}{4}-6p+\frac{9}{4}\left(1+\frac{10}{p}\right)r=f(r).$$

If $p\geq 4$, we have $f(r)\geq f(0)=12+\dfrac{(p-4)(9p+12)}{4}\geq 12.$

If $3\leq p\leq 4$, we have $f(r)\overset{Schur}{\geq}f\left(\dfrac{4pq-p^3}{9}\right)\overset{q=p}{=}12+\dfrac{(p-3)(4-p)(p+4)}{4}\geq 12.$

Equality holds iff $a=b=c=1$ & $a=b=2, c=0$ and its permutations.

1782. **If $a,b,c>0,\ a+b+c=3$ then:**

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma:

$$\frac{t}{2\sqrt{t}+t}\leq\frac{2+t}{9}\ \forall\, t>0$$

Proof:

$$\frac{t}{2\sqrt{t}+t}\leq\frac{2+t}{9}\ \ or\ (2+t)\left(2\sqrt{t}+t\right)\geq 9t\ \ or\ (2+m^2)(2m+m^2)\overset{t=m^2}{\geq}9m^2$$

$$or\ m^4+2m^3-7m^2+4m\geq 0\ or\ m(m-1)^2(m+4)\geq 0\ true$$

$$\frac{a^2(b+1)}{a+b+ab}+\frac{b^2(c+1)}{b+c+bc}+\frac{c^2(a+1)}{c+a+ca}=\sum\frac{a^2(b+1)}{a+b+ab}=$$

$$=\sum\frac{a^2(b+1)}{a(b+1)+b}=\sum\left(a-\frac{ab}{a(b+1)+b}\right)=$$

$$=\sum a-\sum\frac{ab}{ab+a+b}\overset{AM-GM}{\geq}\sum a-\sum\frac{ab}{ab+2\sqrt{ab}}\overset{Lemma}{\geq}$$

$$\sum a-\sum\frac{2+ab}{9}\overset{a+b+c=3}{=}3-\frac{6+ab+bc+ca}{9}\geq$$

$$\geq 3-\frac{6+\frac{(a+b+c)^2}{3}}{9}=3-\frac{6+\frac{9}{3}}{9}=3-1=2$$

Equality holds for $a=b=c=1$.

1783. Let $a,b,c\geq 0$ such that $a+b+c=5$ and $ab+bc+ca>0$. Prove that :

$$\frac{a\sqrt{a+bc}}{b+c}+\frac{b\sqrt{b+ca}}{c+a}+\frac{c\sqrt{c+ab}}{a+b}\geq\sqrt{10}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

$$\sum_{cyc}\frac{a\sqrt{a+bc}}{b+c}=2\sqrt{10}.\sum_{cyc}\frac{a(a+bc)}{2.\sqrt{2}(b+c).\sqrt{5(a+bc)}}\geq 2\sqrt{10}.\sum_{cyc}\frac{a(a+bc)}{2(b+c)^2+5(a+bc)}$$

$$=\sqrt{10}+\sqrt{10}.\sum_{cyc}a.\frac{5(a+bc)-2(b+c)^2}{5[2(b+c)^2+5(a+bc)]}=\sqrt{10}+\frac{\sqrt{10}}{5}.M.$$

So it suffices to prove that $M\geq 0$. WLOG, we assume that $a\geq b\geq c$.

Let $A=2(b+c)^2+5(a+bc), B=2(c+a)^2+5(b+ca), C$
$=2(a+b)^2+5(c+ab)$.

Since $C-B=(b-c)(5+7a)\geq 0$ and $B-A=(a-b)(5+7c)$
≥ 0, then we have $A\leq B\leq C$,

and since $5(c+ab)-2(a+b)^2=(a+b+c)c+ab-2a^2-2b^2\leq 0$, and

$$5(a+bc)-2(b+c)^2=(a+b+c)a+bc-2b^2-2c^2$$
$$=(a-b)(a+2b)+c(a+b-c)\geq 0,$$

Then

$$M=\frac{a[5(a+bc)-2(b+c)^2]}{A}+\frac{b[5(b+ca)-2(c+a)^2]}{B}+\frac{c[5(c+ab)-2(a+b)^2]}{C}$$

$$\overset{A\geq B\geq C}{\geq}\frac{a[5(a+bc)-2(b+c)^2]}{B}+\frac{b[5(b+ca)-2(c+a)^2]}{B}+\frac{c[5(c+ab)-2(a+b)^2]}{B}$$

$$=\frac{a^3+b^3+c^3+3abc-a^2(b+c)-b^2(c+a)-c^2(a+b)}{B}\overset{Schur}{\geq}0.$$

Equality holds iff $\left(a=b=c=\frac{5}{3}\right)$ **and** $\left(a=b=\frac{5}{2},c=0\right)$ **and its permutations.**

1784. If $a,b,c>0$ then:

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{(a+b+c)^2}{2\sqrt{3(ab+bc+ca)}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(ab+bc+ca)=\sqrt{(ab+bc+ca)^2}=\sqrt{(ab+bc+ca)(ab+bc+ca)}\leq$$

$$\leq\sqrt{\frac{(ab+bc+ca)(a+b+c)^2}{3}}=\frac{a+b+c}{\sqrt{3}}\sqrt{ab+bc+ca}\ (1)$$

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=\sum\frac{a^2}{b+c}=\sum\frac{a^3}{ab+ac}\overset{Holder}{\geq}\frac{(a+b+c)^3}{3(2ab+2bc+2ca)}\overset{(1)}{\geq}$$

$$\geq\frac{(a+b+c)^3}{6\cdot\frac{a+b+c}{\sqrt{3}}\sqrt{ab+bc+ca}}\geq\frac{(a+b+c)^2}{2\sqrt{3(ab+bc+ca)}}$$

Equality holds for $a=b=c=1$.

1785. Let $a,b,c\geq 0, ab+bc+ca>0$, then prove that :

$$\sqrt{\frac{a(b+c)}{(2b+c)(2c+b)}}+\sqrt{\frac{b(c+a)}{(2c+a)(2a+c)}}+\sqrt{\frac{c(a+b)}{(2a+b)(2b+a)}}\geq\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $a=x^2, b=y^2, c=z^2$, where $x,y,z\geq 0$. By $GM-HM$ inequality, we have

$$\frac{(2b+c)(2c+b)}{b+c}=2(b+c)+\frac{bc}{b+c}\leq 2(b+c)+\frac{\sqrt{bc}}{2}$$
$$=\frac{4(y^2+z^2)+yz}{2}\ \text{(and analogs)}.$$

Then

$$\sum_{cyc}\sqrt{\frac{a(b+c)}{(2b+c)(2c+b)}}\geq\sqrt{2}\sum_{cyc}\frac{x}{\sqrt{4(y^2+z^2)+yz}}\overset{H\ddot{o}lder}{\geq}\sqrt{\frac{2(x+y+z)^3}{\sum_{cyc}x(4y^2+yz+4z^2)}}\overset{?}{\geq}\sqrt{2}$$

$$\Leftrightarrow x^3+y^3+z^3+3xyz\geq x^2(y+z)+y^2(z+x)+z^2(x+y),$$

which is Schur's inequality. So the proof is complete.

Equality holds iff $(a=b=c)$ and $(a=b>0, c=0)$ and its permutations.

1786. If $x,y,z\in[0,1]$ then:

$$\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}-3xyz\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be:

$$f:[0,1]X[0,1]X[0,1]\to\mathbb{R}, f(x,y,z)=\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}-3xyz$$

$$f'_x=\frac{1}{(2-x)^2}-3yz,\qquad f''(x)=\frac{2}{(2-x)^3}>0$$

Analogous:

$$f''(y)>0, f''(z)>0.$$

f −convex in each variable on the cube $[0,1]X[0,1]X[0,1]$.

$$minf(x,y,z)=min\begin{Bmatrix}f(0,0,0),f(1,0,0),f(0,1,0),f(0,0,1),f(0,1,0),\\f(0,0,1),f(1,1,0),f(1,0,1),f(0,1,1),f(1,1,1)\end{Bmatrix}=$$

$$=min\left\{\frac{3}{2},2,\frac{5}{2},0\right\}=0\Rightarrow minf(x,y,z)=0\Rightarrow$$

$$\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}-3xyz\geq 0$$

1787. **If $a,b,c>0$ and $ab+bc+ca=1$, then prove that :**

$$\frac{(a^2+1)^2}{bc(b+c)}+\frac{(b^2+1)^2}{ca(c+a)}+\frac{(c^2+1)^2}{ab(a+b)}\geq 8\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $\sum_{cyc}a=s\Rightarrow a=s-x, b=s-y, c=s-z\to(1)$,

$$abc=r^2s\to(2),\sum_{cyc}ab=4Rr+r^2\to(3)$$

$$\text{Now, }\frac{(a^2+1)^2}{bc(b+c)}+\frac{(b^2+1)^2}{ca(c+a)}+\frac{(c^2+1)^2}{ab(a+b)}\overset{ab+bc+ca=1}{=}\sum_{cyc}\frac{a(a^2+ab+bc+ca)^2}{abc(b+c)}$$

$$=\sum_{cyc}\frac{a(a+b)^3(c+a)^3}{abc(a+b)(b+c)(c+a)}\overset{\substack{\text{via (1) and (2)}\\ \text{and}\\ \because ab+bc+ca=1}}{=}\sum_{cyc}\frac{y^3z^3(s-x)}{r^2s.4Rrs}\cdot\frac{1}{\sqrt{\sum_{cyc}ab}}$$

$$\overset{\text{via (3)}}{=}\frac{1}{4Rr^3s^2.\sqrt{4Rr+r^2}}\cdot\left(s\sum_{cyc}x^3y^3-4Rrs\sum_{cyc}x^2y^2\right)$$

$$\overset{\text{Gerretsen + Euler}}{\geq}\frac{1}{4Rr^3s.\sqrt{\frac{s^2}{3}}}\cdot\begin{pmatrix}(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\\-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)\end{pmatrix}\overset{?}{\geq}8\sqrt{3}$$

$$\Leftrightarrow(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)$$

$$-32Rr^3s^2 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because P = s^4(s^2-16Rr+5r^2)-2r^2s^2(s^2-4R^2-4Rr-3r^2)$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (24R^2-48Rr-3r^2)s^2+r^2(4R+r)^2 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and } (**) \text{ is trivially true if :}$$

$$24R^2-48Rr-3r^2 \geq 0 \text{ and so, we now consider the case when :}$$

$$24R^2-48Rr-3r^2 < 0; \text{and then : LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(24R^2-48Rr-3r^2)(4R^2+4Rr+3r^2)+r^2(4R+r)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 24t^4-24t^3-29t^2-37t-2 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(24t^3+24t^2+19t+1) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{(a^2+1)^2}{bc(b+c)}+\frac{(b^2+1)^2}{ca(c+a)}+\frac{(c^2+1)^2}{ab(a+b)} \geq 8\sqrt{3}\ \forall\, a,b,c>0 \,\middle|\, ab+bc+ca=1,$$

$$''='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}} \text{ (QED)}$$

1788. **Let $a,b,c \geq 0, a+b+c=3.$ Prove that :**

$$\frac{ab-1}{2a+2b-11}+\frac{bc-1}{2b+2c-11}+\frac{ca-1}{2c+2a-11} \geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c=3, q := ab+bc+ca, r := abc.$ The desired inequality is equivalent to

$$\sum_{cyc}\frac{bc-1}{2b+2c-11} \geq 0 \Leftrightarrow \sum_{cyc}\frac{1-bc}{11-2(3-a)} \geq 0 \Leftrightarrow \sum_{cyc}\frac{1-bc}{5+2a} \geq 0$$

$$\Leftrightarrow \sum_{cyc}(1-bc)(5+2b)(5+2c) \geq 0$$

$$\Leftrightarrow 75+20\sum_{cyc}a-21\sum_{cyc}bc-10\sum_{cyc}bc(b+c)-4\sum_{cyc}(bc)^2 \geq 0$$

$$\Leftrightarrow 135-51q-4q^2+54r \geq 0.\ (1)$$

Since $q \leq \frac{p^2}{3}=3$, then we have two cases:

If $q \le \frac{9}{4}$, we have : $LHS_{(1)} \ge 135 - 51q - 4q^2 = (9-4q)(15+q) \ge 0$.

If $\frac{9}{4} \le q \le 3$, we have :

$$LHS_{(1)} \overset{Schur}{\gtrsim} 135 - 51q - 4q^2 + 6(12q-27) = (3-q)(4q-9) \ge 0.$$

Equality holds iff $a=b=c=1$ & $a=b=\frac{3}{2}, c=0$ and permutations.

1789. If $x,y,z>0,\ xy\sqrt{xy}+yz\sqrt{yz}+zx\sqrt{zx}=1$ then:

$$\frac{x^6}{x^3+y^3}+\frac{y^6}{y^3+z^3}+\frac{z^6}{z^3+x^3} \ge \frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1 = xy\sqrt{xy}+yz\sqrt{yz}+zx\sqrt{zx} =$$

$$\sum xy\sqrt{xy} = \sum \sqrt{x^3y^3} \overset{CBS}{\le} \sqrt{3(x^3y^3+y^3z^3+z^3x^3)} \le$$

$$\le \sqrt{\frac{3(x^3+y^3+z^3)^2}{3}} = x^3+y^3+z^3 \ (1)$$

$$\frac{x^6}{x^3+y^3}+\frac{y^6}{y^3+z^3}+\frac{z^6}{z^3+x^3} = \sum \frac{x^6}{x^3+y^3} =$$

$$= \sum\left(x^3 - \frac{x^3y^3}{x^3+y^3}\right) = \sum x^3 - \sum \frac{x^3y^3}{x^3+y^3} \ge$$

$$\overset{AM-GM}{\ge} \sum x^3 - \sum \frac{x^3y^3}{2\sqrt{x^3y^3}} = \sum x^3 - \frac{1}{2}\sum\sqrt{x^3y^3} \overset{(1)}{\gtrsim} 1 - \frac{1}{2} = \frac{1}{2}$$

$$\textit{Equality holds for } x=y=z=\frac{1}{\sqrt[3]{3}}$$

1790. Let $a, b, c \geq 0, ab + bc + ca = 2$. Prove that :

$$\frac{a+b+c}{3} \geq \sqrt{\frac{a^2b + b^2c + c^2a + 2}{6}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By $\text{AM} - \text{GM}$ inequality, we have

$$4ab^2 + a(2-c)^2 \geq 4ab(2-c) \text{ (and analogs)}$$

Adding this inequality with the similar ones, we get

$$4(a+b+c) + 4(ab^2 + bc^2 + ca^2) + (a^2b + b^2c + c^2a) \geq 24 - 12abc$$

$$\Leftrightarrow 4(a+b+c) + 4(a+b+c)(ab+bc+ca) \geq 3(a^2b + b^2c + c^2a) + 24$$

$$\Leftrightarrow 12(a+b+c) \geq 3(a^2b + b^2c + c^2a) + 24$$

$$\Rightarrow 2(a+b+c)^2 + 18 \overset{\text{AM-GM}}{\geq} 12(a+b+c) \geq 3(a^2b + b^2c + c^2a) + 24$$

$$\Rightarrow \frac{a+b+c}{3} \geq \sqrt{\frac{a^2b + b^2c + c^2a + 2}{6}}.$$

Equality holds iff $(a, b, c) = (2, 1, 0)$ and its cyclic permutations.

1791. If $a, b, c > 0$, then prove that :

$$\frac{1}{2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0,$
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius
$= s, R, r$ (say); and then : $\sum_{cyc} a = s \rightarrow (1), \sum_{cyc} ab = 4Rr + r^2 \rightarrow (2)$ and

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3)\ \&\ now, \frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

$$\Leftrightarrow \frac{7}{2} + \frac{\sum_{cyc} a^2}{\sum_{cyc} ab} \ge \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{b+c}\right) \overset{\text{via } (1),(2) \text{ and } (3)}{\Leftrightarrow} \frac{7}{2} + \frac{s^2 - 8Rr - 2r^2}{4Rr + r^2}$$

$$\ge s.\frac{s^2 + 4Rr + r^2}{4Rrs} \Leftrightarrow s^2 - (8R^2 - 2Rr - r^2) \le 0 \Leftrightarrow$$

$$\left(s^2 - (4R^2 + 4Rr + 3r^2)\right) - 2(R - 2r)(2R + r) \le 0 \to \text{true via Gerretsen \& Euler}$$

$$\therefore \frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \forall a,b,c > 0, \text{"="} \text{ iff } a = b = c \text{ (QED)}$$

1792. Let a, b, c be real numbers. Prove that :

$$\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \ge \frac{4(a^2 + b^2 + c^2 + ab + bc + ca)}{(a^2 + b^2 + c^2)^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := a^2 + b^2 + c^2, y := ab + bc + ca$.

By CBS inequality, we have

$$\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} \ge \frac{9}{2(a^2 + b^2 + c^2) - (ab + bc + ca)} =$$

$$= \frac{9}{2x - y} \overset{?}{\ge} \frac{4(x + y)}{x^2} \Leftrightarrow (x - 2y)^2 \ge 0,$$

which is true. Equality holds iff $a = b, c = 0$ and its permutation.

1793. If $x, y > 0,\ xy = 1$ then:

$$\frac{x^4}{y^3 + 1} + \frac{y^4}{x^3 + 1} \ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(x^7 + y^7) + (x^4 + y^4) = (x^{4+3} + y^{4+3}) + (x^4 + y^4) \overset{Chebyshev\ \&\ AM-GM}{\ge}$$

$$\geq \frac{1}{2}(x^3+y^3)(x^4+y^4)+2(x^4y^4)^{\frac{1}{2}} \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{2}(x^3+y^3)2(x^4y^4)^{\frac{1}{2}}+2(x^4y^4)^{\frac{1}{2}} \overset{xy=1}{=} (x^3+y^3)+2 \ (1)$$

$$\frac{x^4}{y^3+1}+\frac{y^4}{x^3+1}=\frac{(x^7+y^7)+(x^4+y^4)}{x^3y^3+(x^3+y^3)+1} \overset{xy=1}{=}$$

$$=\frac{(x^7+y^7)+(x^4+y^4)}{2+(x^3+y^3)} \overset{(1)}{\geq} \frac{(x^3+y^3)+2}{(x^3+y^3)+2}=1$$

Equality holds for $x=y=1$.

1794. Let $a,b,c \geq 0, ab+bc+ca=1$. Prove that :

$$\frac{a}{\sqrt{a+1}}+\frac{b}{\sqrt{b+1}}+\frac{c}{\sqrt{c+1}} \geq \frac{2\sqrt{3}}{\sqrt{a+b+c+3abc+4}}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\frac{a}{\sqrt{a+1}}+\frac{b}{\sqrt{b+1}}+\frac{c}{\sqrt{c+1}} \geq \sqrt{\frac{(a+b+c)^3}{a(a+1)+b(b+1)+c(c+1)}}.$$

So it suffices to prove that

$$(a+b+c)^3(a+b+c+3abc+4) \geq 12(a^2+b^2+c^2+a+b+c),$$

Let $p \coloneqq a+b+c$, $q \coloneqq ab+bc+ca$, $r \coloneqq abc$. The last inequality is equivalent to

$$f(r)=p^4+4p^3-12p^2-12p+24+3p^3r \geq 0.$$

We have $p \geq \sqrt{3q}=\sqrt{3}$, and by the fourth degree Schur's inequality, we have

$$r \geq \frac{(p^2-q)(4q-p^2)}{6p}=\frac{(p^2-1)(4-p^2)}{6p}.$$

If $p \geq 2$, we have

$$f(r) \geq f(0) = p^4 + 4p^3 - 12p^2 - 12p + 24 = (p-2)(p^3 + 6p^2 - 12) \geq 0.$$

If $\sqrt{3} \leq p \leq 2$, we have $f(r) \geq p^4 + 4p^3 - 12p^2 - 12p + 24 + \dfrac{p^2(p^2-1)(4-p^2)}{2}$

$$= \frac{1}{2}(2-p)(p-\sqrt{3})(p^4 + (2+\sqrt{3})p^3 + 2\sqrt{3}p^2 - 8p - 8\sqrt{3}) \geq 0.$$

Equality holds iff $a = b = c = \dfrac{\sqrt{3}}{3}$ & $a = b = 1, c = 0$ **and its permutation.**

1795. **If $a, b, c > 0$, $a^2b^2 + b^2c^2 + c^2a^2 = 12$ then:**

$$c\sqrt[3]{ab^2} + a\sqrt[3]{bc^2} + b\sqrt[3]{ca^2} \leq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum ab = \sqrt{\left(\sum ab\right)^2} \overset{CBS}{\leq} \sqrt{3\sum a^2b^2} \overset{\sum a^2b^2=12}{=} \sqrt{3 \times 12} = 6(1)$$

$$c\sqrt[3]{ab^2} + a\sqrt[3]{bc^2} + b\sqrt[3]{ca^2} = \sum c\sqrt[3]{ab^2} = \sum \sqrt[3]{ac^3b^2} =$$

$$= \sum \sqrt[3]{ac.bc.bc} \overset{AM-GM}{\leq} \sum \frac{ac+bc+bc}{3} =$$

$$= \frac{2}{3}\sum bc + \frac{1}{3}\sum ac \overset{(1)}{\leq} \frac{2}{3} \times 6 + \frac{1}{3} \times 6 = 6$$

Equality holds for $a = b = c = \sqrt{2}$.

1796. **Let $a, b, c \geq 0$, $ab + bc + ca = 1$. Prove that :**

$$\sqrt{a+b-2ab} + \sqrt{b+c-2bc} + \sqrt{c+a-2ca} \geq 2\sqrt{a+b+c-1}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\left(\sqrt{a+b-2ab}+\sqrt{b+c-2bc}+\sqrt{c+a-2ca}\right)^2 \geq \frac{(2a+2b+2c-2)^3}{(a+b-2ab)^2+(b+c-2bc)^2+(c+a-2ca)^2},$$

So it suffices to prove that

$$2(a+b+c-1)^2 \geq (a+b-2ab)^2+(b+c-2bc)^2+(c+a-2ca)^2,$$

which is equivalent to

$$abc[2(a+b+c)-3] \geq 0,$$

which is true since $a+b+c \geq \sqrt{3(ab+bc+ca)} = \sqrt{3} > \frac{3}{2}$.

Equality holds iff $a=b=1, c=0$ **and its permutation.**

1797. If $a,b,c>0, a+b+c=\frac{3}{2}$ then:

$$\frac{1+b}{1+4a^2}+\frac{1+c}{1+4b^2}+\frac{1+a}{1+4c^2} \geq \frac{9}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1+b}{1+4a^2} = (1+b) - \frac{4a^2(1+b)}{1+4a^2} \overset{AM-GM}{\geq} (1+b) - \frac{4a^2(1+b)}{4a} =$$

$$= (1+b) - a(1+b) = 1+b-a-ab \ (1)$$

$$\frac{1+b}{1+4a^2}+\frac{1+c}{1+4b^2}+\frac{1+a}{1+4c^2} = \sum \frac{1+b}{1+4a^2} \overset{(1)}{\geq} \sum(1+b-a-ab)$$

$$= 3+\sum b - \sum a - \sum ab \geq 3 - \frac{(\sum a)^2}{3} = 3 - \frac{\left(\frac{3}{2}\right)^2}{3} = 3 - \frac{3}{4} = \frac{9}{4}$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{2}.$$

1798. Let $a,b,c\geq 0, a+b+c=3.$ Prove that :

$$\sqrt{a^2+2abc+b^2}+\sqrt{b^2+2abc+c^2}+\sqrt{c^2+2abc+a^2}\leq 6$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG we assume that $a=max\{a,b,c\}$. We have $1\leq a\leq 3$.

By CBS inequality, we have

$$\sqrt{a^2+2abc+b^2}+\sqrt{a^2+2abc+c^2}\leq\sqrt{2(2a^2+4abc+b^2+c^2)}$$

$$=\sqrt{4a^2+(2a+1)(b+c)^2-(2a-1)(b-c)^2}\leq\sqrt{4a^2+(2a+1)(3-a)^2}$$

$$=\sqrt{2a^3-7a^2+12a+9}.$$

Also, we have

$$2(b^2+2abc+c^2)=(a+1)(b+c)^2-(a-1)(b-c)^2\leq(a+1)(3-a)^2.$$

Let $x:=\sqrt{\frac{a+1}{2}}$. Using the last results, we only need to prove that

$$\sqrt{2a^3-7a^2+12a+9}+(3-a)\sqrt{\frac{a+1}{2}}\leq 6$$

$$\overset{a=2x^2-1}{\Leftrightarrow}\sqrt{16x^6-52x^4+64x^2-12}\leq 6-4x+2x^3$$

$$\overset{squaring}{\Leftrightarrow}\quad 0\leq 4-4x-4x^2+2x^3+3x^4-x^6$$

$$\Leftrightarrow 0\leq(1-x)^2(2-x^2)(2+2x+x^2),$$

which is true since $1\leq x\leq\sqrt{2}$, and the proof is complete.

Equality holds iff $a=b=c=1$ and $a=3, b=c=0$ and its permutation.

1799. If $a,b,c>0,\ a^2+b^2+c^2=1$ then:

$$\frac{\sqrt{3a^2+4ab+3b^2}}{ab}+\frac{\sqrt{3b^2+4bc+3c^2}}{bc}+\frac{\sqrt{3c^2+4ca+3a^2}}{ca}\ge 3\sqrt{30}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3a^2+4ab+3b^2=3(a^2+b^2)+4ab\overset{AM-GM}{\ge}6ab+4ab=10ab\ (1)$$

$$\frac{\sqrt{3a^2+4ab+3b^2}}{ab}+\frac{\sqrt{3b^2+4bc+3c^2}}{bc}+\frac{\sqrt{3c^2+4ca+3a^2}}{ca}=$$

$$=\sum\frac{\sqrt{3a^2+4ab+3b^2}}{ab}\overset{(1)}{\ge}\sum\frac{\sqrt{10ab}}{ab}=$$

$$=\sqrt{10}\sum\frac{1}{\sqrt{ab}}\overset{CBS}{\ge}\frac{\sqrt{10}\,(1+1+1)^2}{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}\overset{CBS}{\ge}\frac{9\sqrt{10}}{\sqrt{3(ab+bc+ca)}}\ge$$

$$\ge\frac{9\sqrt{10}}{\sqrt{3(a^2+b^2+c^2)}}\overset{a^2+b^2+c^2=1}{=}\frac{9\sqrt{10}}{\sqrt{3}}=3\sqrt{30}$$

Equality holds for $a=b=c=1$.

1800. Let $a,b,c\ge 0, ab+bc+ca>0$ and $a+b+c=3$. Prove that :

$$\frac{1}{ab+3c}+\frac{1}{bc+3a}+\frac{1}{ca+3b}\ge\frac{24}{5abc+27}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1}{ab+3c}+\frac{1}{bc+3a}+\frac{1}{ca+3b}=$$

$$=\frac{1}{ab+c(a+b+c)}+\frac{1}{bc+a(a+b+c)}+\frac{1}{ca+b(a+b+c)}=$$

$$=\frac{1}{(c+a)(c+b)}+\frac{1}{(a+b)(a+c)}+\frac{1}{(b+c)(b+a)}=\frac{2(a+b+c)}{(a+b)(b+c)(c+a)}$$

$$=\frac{24}{4(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)+8abc}\overset{Schur}{\geq}\frac{24}{(a+b+c)^3+5abc}$$

$$=\frac{24}{5abc+27}.$$

Equality holds iff $a=b=c=1$ and $a=b=\frac{3}{2}, c=0$ and its permutation.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru