ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\sqrt{\sin A + \sin B}}{\sin^2 A + \sin^2 B}\left(\frac{1}{\sqrt{\sin B}} + \frac{1}{\sqrt{\sin C}}\right) \ge 4\sqrt{2}$$

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Solution by Tapas Das-India

$$\sqrt[3]{(\sin^2 A + \sin^2 B)(\sin^2 B + \sin^2 C)(\sin^2 C + \sin^2 A)} \overset{AM-GM}{\le} \frac{2\sum \sin^2 A}{3} =$$

$$= \frac{2}{3}\frac{(a^2 + b^2 + c^2)}{4R^2} \overset{Leibniz}{\le} \frac{2}{3}\cdot\frac{9R^2}{4R^2} = \frac{3}{2} \ (1)$$

$$\frac{1}{\sqrt{\sin C}} + \frac{1}{\sqrt{\sin B}} = \frac{1^{\frac{3}{2}}}{\sqrt{\sin C}} + \frac{1^{\frac{3}{2}}}{\sqrt{\sin B}} \overset{Radon}{\ge} \frac{2^{\frac{3}{2}}}{\sqrt{\sin B + \sin C}} \quad (2)$$

$$\sum \frac{\sqrt{\sin A + \sin B}}{\sin^2 A + \sin^2 B}\left(\frac{1}{\sqrt{\sin B}} + \frac{1}{\sqrt{\sin C}}\right) \overset{(2)}{\ge} 2^{\frac{3}{2}} \sum \frac{\sqrt{\frac{\sin A + \sin B}{\sin B + \sin C}}}{\sin^2 A + \sin^2 B} \overset{AM-GM}{\ge}$$

$$\ge 3 \times 2\sqrt{2}\frac{1}{\sqrt[3]{(\sin^2 A + \sin^2 B)(\sin^2 B + \sin^2 C)(\sin^2 C + \sin^2 A)}} \overset{(1)}{\ge} \frac{6\sqrt{2}}{\frac{3}{2}} = 4\sqrt{2}$$

Equality holds for an equilateral triangle.