ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{m_a^4\sqrt{m_a^2+m_c^2}+m_b^4\sqrt{m_b^2+m_c^2}}{m_a^2+m_b^2} \geq 81\sqrt{2}r^3$$

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Solution by Tapas Das-India

$$m_a^4+m_b^4=(m_a^2)^2+\left(m_b^2\right)^2 \overset{CBS}{\geq} \frac{1}{2}\left(m_a^2+m_b^2\right)^2=$$
$$=\frac{1}{2}\left(m_a^2+m_b^2\right)\left(m_a^2+m_b^2\right) \overset{CBS}{\geq} \frac{1}{2}\left(m_a^2+m_b^2\right)\frac{1}{2}(m_a+m_b)^2=$$
$$=\frac{1}{4}\left(m_a^2+m_b^2\right)(m_a+m_b)^2 \ (1)$$

$$\sqrt{m_a^2+m_c^2}+\sqrt{m_b^2+m_c^2}=\sqrt{m_a^2+m_c^2}+\sqrt{m_c^2+m_b^2} \overset{Minkowski}{\geq}$$
$$\geq \sqrt{(m_a+m_c)^2+(m_c+m_b)^2} \overset{AM-GM}{\geq} \sqrt{2(m_a+m_c)(m_b+m_c)} \ (2)$$

$$WLOG\ a \geq b \geq c\ then\ m_a \leq m_b \leq m_c\ and$$
$$\left(m_a^2+m_b^2\right) \leq \left(m_a^2+m_c^2\right) \leq \left(m_c^2+m_b^2\right)$$

$$\sum \frac{m_a^4\sqrt{m_a^2+m_c^2}+m_b^4\sqrt{m_b^2+m_c^2}}{m_a^2+m_b^2} \overset{Chebyshev}{\geq}$$
$$\geq \sum \frac{\frac{1}{2}\left(m_a^4+m_b^4\right)\left(\sqrt{m_a^2+m_c^2}+\sqrt{m_b^2+m_c^2}\right)}{\left(m_a^2+m_b^2\right)} \overset{(1)\&(2)}{\geq}$$
$$\geq \frac{1}{8}\sum (m_a+m_b)^2\sqrt{2(m_a+m_c)(m_b+m_c)} \overset{AM-GM}{\geq}$$
$$\geq \frac{3\sqrt{2}}{8}\sqrt[3]{(m_a+m_b)^3.(m_b+m_c)^3(m_c+m_a)^3}=$$
$$=\frac{3\sqrt{2}}{8}(m_a+m_b)(m_b+m_c)(m_c+m_a) \overset{Cesaro}{\geq} \frac{3\sqrt{2}}{8}.8m_am_bm_c \overset{m_a\geq\sqrt{s(s-a)}}{\geq}$$
$$\geq \frac{3\sqrt{2}}{8}.8s^2r \overset{Mitrinovic}{\geq} 3\sqrt{2}.27r^3=81\sqrt{2}r^3$$

Equality holds for an equilateral triangle.