ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a+r_b}\left(\sqrt{\frac{r_a^2+r_b^2}{r_b^2+r_c^2}}+\sqrt{\frac{r_a^2+r_b^2}{r_c^2+r_a^2}}\right) \ge \frac{2}{\sqrt{3R^2-8r^2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$r_a^2+r_b^2=\frac{r_a^2}{1}+\frac{r_b^2}{1}\overset{Radon}{\ge}\frac{(r_a+r_b)^2}{2}\ (1)$$

$$r_a^2+r_b^2+r_c^2=(r_a+r_b+r_c)^2-2(r_ar_b+r_br_c+r_cr_a)=$$

$$=(4R+r)^2-2s^2\overset{Euler\ \&\ Mitrinovic}{\le}\left(\frac{9R}{2}\right)^2-54r^2=\frac{27}{4}(3R^2-8r^2)\ (A)$$

$$\frac{1}{r_a+r_b}\left(\sqrt{\frac{r_a^2+r_b^2}{r_b^2+r_c^2}}+\sqrt{\frac{r_a^2+r_b^2}{r_c^2+r_a^2}}\right)=\frac{\sqrt{r_a^2+r_b^2}}{r_a+r_b}\left(\frac{1}{\sqrt{r_b^2+r_c^2}}+\frac{1}{\sqrt{r_c^2+r_a^2}}\right)\overset{(1)}{\ge}$$

$$\ge\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{r_b^2+r_c^2}}+\frac{1}{\sqrt{r_c^2+r_a^2}}\right)\ (2)$$

$$\sum\frac{1}{r_a+r_b}\left(\sqrt{\frac{r_a^2+r_b^2}{r_b^2+r_c^2}}+\sqrt{\frac{r_a^2+r_b^2}{r_c^2+r_a^2}}\right)\overset{(2)}{\ge}\sum\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{r_b^2+r_c^2}}+\frac{1}{\sqrt{r_c^2+r_a^2}}\right)$$

$$=\frac{2}{\sqrt{2}}\sum\left(\frac{1}{\sqrt{r_b^2+r_c^2}}\right)=\sqrt{2}\sum\left(\frac{1^{\frac{3}{2}}}{\sqrt{r_b^2+r_c^2}}\right)\overset{Radon}{\ge}$$

$$\ge\frac{\sqrt{2}\times 3\sqrt{3}}{\sqrt{2(r_a^2+r_b^2+r_c^2)}}\overset{(A)}{\ge}\frac{2}{\sqrt{3R^2-8r^2}}$$

Equality holds for an equilateral triangle.