ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationships holds:

$$1)\ m_a^5+m_b^5+m_c^5\le\frac{3^6(81R^5-2560r^5)}{32}$$

$$2)\ r_a^5+r_b^5+r_c^5\le\frac{3^6(81R^5-2560r^5)}{32}$$

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$$\textit{Well known result}:$$

$$(x+y+z)^5=x^5+y^5+z^5+5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$$

$$\sum m_a\overset{Gotman\ II}{\le}\frac{9R}{2}\ \ (1)$$

$$\prod(m_a+m_b)\overset{Cesaro}{\ge}8m_am_bm_c\overset{m_a\ge\sqrt{s(s-a)}}{\ge}8s^2r\overset{Mitinovic}{\ge}8\cdot3^3\cdot r^3\ \ (2)$$

$$\sum m_a^2+\sum m_am_b\ge2\sum m_am_b\overset{m_a\ge h_a}{\ge}2\sum h_ah_b=\frac{4s^2r}{R}\overset{cosnita\ \&\ Trtoiu}{\ge}2.3^3r^2\ \ (3)$$

$$m_a^5+m_b^5+m_c^5=\left(\sum m_a\right)^3-5\prod(m_a+m_b)\times\left(\sum m_a^2+\sum m_am_b\right)\overset{(1),(2)\&(3)}{\le}$$

$$\le\left(\frac{9R}{2}\right)^5-5.8.3^3.r^3.2.3^3r^2=\frac{3^{10}R^5}{2^5}-80\times3^6r^5=$$

$$=\frac{3^{10}R^5-2^5.80.3^6r^5}{32}=\frac{3^6(81R^5-2560r^5)}{32}$$

$$r_a^5+r_b^5+r_c^5=\left(\sum r_a\right)^3-5\prod(r_a+r_b)\times\left(\sum r_a^2+\sum r_ar_b\right)\le$$

$$\overset{Cesaro}{\le}(4R+r)^5-5.8\,r_ar_br_c\left(\sum r_ar_b+\sum r_ar_b\right)\overset{Euler}{\le}$$

$$\le\left(\frac{9R}{2}\right)^5-40s^2r\,.2s^2\overset{Mitrinovic}{\le}\left(\frac{9R}{2}\right)^5-80.3^6r^5=$$

$$=\frac{3^{10}R^5-2^5.80.3^6r^5}{32}=\frac{3^6(81R^5-2560r^5)}{32}$$

Equality holds for an equilateral triangle.