ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{m_a+m_b}\left(\sqrt{\frac{m_a^2+m_b^2}{m_b^2+m_c^2}}+\sqrt{\frac{m_a^2+m_b^2}{m_c^2+m_a^2}}\right)\ge \frac{2}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$m_a^2+m_b^2=\frac{m_a^2}{1}+\frac{m_b^2}{1}\overset{Radon}{\ge}\frac{(m_a+m_b)^2}{2}\ (1)$$

$$\frac{1}{m_a+m_b}\left(\sqrt{\frac{m_a^2+m_b^2}{m_b^2+m_c^2}}+\sqrt{\frac{m_a^2+m_b^2}{m_c^2+m_a^2}}\right)=\frac{\sqrt{m_a^2+m_b^2}}{m_a+m_b}\left(\frac{1}{\sqrt{m_b^2+m_c^2}}+\frac{1}{\sqrt{m_c^2+m_a^2}}\right)\overset{(1)}{\ge}$$

$$\ge\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m_b^2+m_c^2}}+\frac{1}{\sqrt{m_c^2+m_a^2}}\right)\ (2)$$

$$\sum\left(\frac{1}{m_a+m_b}\left(\sqrt{\frac{m_a^2+m_b^2}{m_b^2+m_c^2}}+\sqrt{\frac{m_a^2+m_b^2}{m_c^2+m_a^2}}\right)\right)\overset{(2)}{\ge}\sum\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{m_b^2+m_c^2}}+\frac{1}{\sqrt{m_c^2+m_a^2}}\right)$$

$$=\frac{2}{\sqrt{2}}\sum\left(\frac{1}{\sqrt{m_b^2+m_c^2}}\right)=\sqrt{2}\sum\left(\frac{1^{\frac{3}{2}}}{\sqrt{m_b^2+m_c^2}}\right)\overset{Radon}{\ge}\frac{\sqrt{2}\times 3\sqrt{3}}{\sqrt{2(m_a^2+m_b^2+m_c^2)}}$$

$$=\frac{3\sqrt{3}}{\sqrt{\frac{3}{4}(a^2+b^2+c^2)}}\overset{Leibniz}{\ge}\frac{3}{\sqrt{\frac{1}{4}9R^2}}=\frac{2}{R}$$

Equality holds for an equilateral triangle.