ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\sin^5\frac{A}{2}+\sin^3\frac{C}{2}\sin^2\frac{A}{2}}{\sin^3\frac{A}{2}\sin^4\frac{B}{2}\sin\frac{C}{2}+\sin^5\frac{C}{2}\sin^2\frac{B}{2}\sin\frac{A}{2}}\geq 24$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$Walter\ Janous\ inequaliy: x',y',z';\ A,B,C>0\ then$$

$$\frac{x'}{y'+z'}(B+C)+\frac{y'}{z'+x'}(C+A)+\frac{z'}{x'+y'}(A+B)\geq\sqrt{3(AB+BC+CA)}\ (1)$$

Lemma: $\forall\ x,y,z>0,\qquad \sum\frac{x^5+x^2z^3}{x^3y^4z+xy^2z^5}\geq\sqrt{3\sum\frac{1}{x^3y^3}}$

Proof:

$$\frac{\left(x^5+x^2z^3\right)}{x^3y^4z+xy^2z^5}=\frac{\left(\frac{x^2}{y^2}\right)}{\frac{y^2}{z^2}+\frac{z^2}{x^2}}\left(\frac{1}{z^3}+\frac{1}{x^3}\right)$$

$$\sum\frac{x^5+x^2z^3}{x^3y^4z+xy^2z^5}=\sum\frac{\left(\frac{x^2}{y^2}\right)}{\frac{y^2}{z^2}+\frac{z^2}{x^2}}\left(\frac{1}{z^3}+\frac{1}{x^3}\right)\overset{(1)}{\geq}\sqrt{3\sum\frac{1}{x^3y^3}}$$

$$We\ take\ x=\sin\frac{A}{2},y=\sin\frac{B}{2},z=\sin\frac{C}{2}$$

$$and\ 3\sum\frac{1}{x^3y^3}\overset{AM-GM}{\geq}\frac{9}{x^2y^2z^2}=9\prod\csc^2\frac{A}{2}=9\times\left(\frac{4R}{r}\right)^2\overset{Euler}{\geq}$$
$$\geq 9\times16\times4=576=24^2\ (2)$$

$$\sum \frac{\sin^5\frac{A}{2}+\sin^3\frac{C}{2}\sin^2\frac{A}{2}}{\sin^3\frac{A}{2}\sin^4\frac{B}{2}\sin\frac{C}{2}+\sin^5\frac{C}{2}\sin^2\frac{B}{2}\sin\frac{A}{2}}=$$
$$=\sum\frac{x^5+x^2z^3}{x^3y^4z+xy^2z^5}\geq\sqrt{3\sum\frac{1}{x^3y^3}}\overset{(2)}{\geq}24$$

Equality holds for an equilateral triangle.