ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{\left(\frac{\cot^7\frac{A}{2}}{\cot^2\frac{C}{2}}+\frac{\cot^6\frac{B}{2}}{\cot\frac{C}{2}}\right)}{\cot^4\frac{A}{2}+\cot^4\frac{B}{2}} \geq 3\sqrt{3}$$

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$$Lemma: If\ a,b,c>0\ \ then\ \frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}+\frac{\frac{b^7}{a^2}+\frac{c^6}{a}}{b^4+c^4}+\frac{\frac{c^7}{b^2}+\frac{a^6}{b}}{c^4+a^4} \geq 3\sqrt[3]{abc}\quad (A)$$

$$Proof: Walter\ Janous\ inequaliy: x,y,z;\ A,B,C>0\ then$$

$$\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B) \geq \sqrt{3(AB+BC+CA)}\ (1)$$

$$\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}=\frac{\frac{1}{c^4}(a^7c^2+b^6c^3)}{a^4+b^4}=\frac{\frac{1}{c^4}\frac{(a^7c^2+b^6c^3)}{a^4b^4}}{\frac{a^4+b^4}{a^4b^4}}=$$

$$=\frac{\left(\frac{1}{c^4}\right)}{\frac{1}{a^4}+\frac{1}{b^4}}\left(\frac{a^3c^2}{b^4}+\frac{b^2c^3}{a^4}\right)=\frac{z}{x+y}(A+B)\ (2)$$

$$\left(x=\frac{1}{a^4},y=\frac{1}{b^4}z=\frac{1}{c^4};\ A=\frac{b^2c^3}{a^4},B=\frac{a^3c^2}{b^4},C=\frac{b^3a^2}{c^4}\right)$$

$$\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}+\frac{\frac{b^7}{a^2}+\frac{c^6}{a}}{b^4+c^4}+\frac{\frac{c^7}{b^2}+\frac{a^6}{b}}{c^4+a^4}=\sum\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}\overset{(2)}{=}\sum\frac{y}{x+z}(C+A)\overset{(1)}{\geq}$$

$$\geq\sqrt{3(AB+BC+CA)}\overset{AM-GM}{\geq}\sqrt{3\times3\sqrt[3]{A^2B^2C^2}}=3\sqrt[3]{ABC}=$$

$$=3\sqrt[3]{\left(\frac{b^2c^3}{a^4}\cdot\frac{a^3c^2}{b^4}\cdot\frac{b^3a^2}{c^4}\right)}=3\sqrt[3]{abc}$$

$$(proof\ complete)$$

$$now,\quad we\ take\ a=\cos\frac{A}{2},b=\cos\frac{B}{2},c=\cos\frac{C}{2}$$

$$\sum\frac{\left(\frac{\cot^7\frac{A}{2}}{\cot^2\frac{C}{2}}+\frac{\cot^6\frac{B}{2}}{\cot\frac{C}{2}}\right)}{\cot^4\frac{A}{2}+\cot^4\frac{B}{2}}=\sum\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}\overset{Lemma}{\geq}3\sqrt[3]{abc}=$$

$$=3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=3\sqrt[3]{\frac{s}{r}}\overset{Mitrinovic}{\geq}3\sqrt{3}$$

Equality holds for A=B=C.