ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\dfrac{\cot^7\frac{A}{2}}{\cot^2\frac{C}{2}}+\dfrac{\cot^6\frac{B}{2}}{\cot\frac{C}{2}}}{\cot^4\frac{A}{2}+\cot^4\frac{B}{2}}\geq 3\sqrt{3}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\geq\sqrt{3\sum_{cyc}A'B'}$$

$$\text{(via Walter Janous)}\to (1) \text{ and now, }\sum_{cyc}\frac{\dfrac{\cot^7\frac{A}{2}}{\cot^2\frac{C}{2}}+\dfrac{\cot^6\frac{B}{2}}{\cot\frac{C}{2}}}{\cot^4\frac{A}{2}+\cot^4\frac{B}{2}}=$$

$$\sum_{cyc}\frac{\dfrac{1}{\cot^4\frac{C}{2}}\left(\dfrac{\cot^2\frac{B}{2}\cot^3\frac{C}{2}}{\cot^4\frac{A}{2}}+\dfrac{\cot^2\frac{C}{2}\cot^3\frac{A}{2}}{\cot^4\frac{B}{2}}\right)}{\dfrac{1}{\cot^4\frac{A}{2}}+\dfrac{1}{\cot^4\frac{B}{2}}}=\sum_{cyc}\frac{\dfrac{1}{\cot^4\frac{A}{2}}\left(\dfrac{\cot^2\frac{C}{2}\cot^3\frac{A}{2}}{\cot^4\frac{B}{2}}+\dfrac{\cot^2\frac{A}{2}\cot^3\frac{B}{2}}{\cot^4\frac{C}{2}}\right)}{\dfrac{1}{\cot^4\frac{B}{2}}+\dfrac{1}{\cot^4\frac{C}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c} x'=\dfrac{1}{\cot^4\frac{A}{2}},\, y'=\dfrac{1}{\cot^4\frac{B}{2}},\, z'=\dfrac{1}{\cot^4\frac{C}{2}}, \\ A'=\dfrac{\cot^2\frac{B}{2}\cot^3\frac{C}{2}}{\cot^4\frac{A}{2}},\, B'=\dfrac{\cot^2\frac{C}{2}\cot^3\frac{A}{2}}{\cot^4\frac{B}{2}},\, C'=\dfrac{\cot^2\frac{A}{2}\cot^3\frac{B}{2}}{\cot^4\frac{C}{2}} \end{array}\right)$$

$$\overset{\text{via }(1)}{\geq}\sqrt{3\sum_{cyc}\left(\frac{\cot^2\frac{B}{2}\cot^3\frac{C}{2}}{\cot^4\frac{A}{2}}\cdot\frac{\cot^2\frac{C}{2}\cot^3\frac{A}{2}}{\cot^4\frac{B}{2}}\right)}\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{\prod_{cyc}\cot^2\frac{A}{2}}}=3\sqrt[6]{\frac{s^2}{r^2}}$$

$$\overset{\text{Mitrinovic}}{\geq}3.\sqrt[6]{27}=3\sqrt{3}\ \forall\,\Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$