ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\frac{\sin^4\frac{B}{2}}{\sin^4\frac{A}{2}}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)+\frac{\sin^4\frac{C}{2}}{\sin^4\frac{A}{2}}\left(\sin^3\frac{B}{2}+\sin^3\frac{C}{2}\right)}{\sin^4\frac{B}{2}+\sin^4\frac{C}{2}}\geq 12$$

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$$\forall\, A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\geq\sqrt{3\sum_{cyc}A'B'}$$

$$\text{(via Walter Janous)}\to (1) \text{ and } \sum_{cyc}\frac{\frac{\sin^4\frac{B}{2}}{\sin^4\frac{A}{2}}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)+\frac{\sin^4\frac{C}{2}}{\sin^4\frac{A}{2}}\left(\sin^3\frac{B}{2}+\sin^3\frac{C}{2}\right)}{\sin^4\frac{B}{2}+\sin^4\frac{C}{2}}$$

$$=\sum_{cyc}\frac{\frac{1}{\sin^4\frac{A}{2}}\cdot\left(\frac{\sin^3\frac{B}{2}+\sin^3\frac{C}{2}}{\sin^4\frac{B}{2}}+\frac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin^4\frac{C}{2}}\right)}{\frac{1}{\sin^4\frac{B}{2}}+\frac{1}{\sin^4\frac{C}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\frac{1}{\sin^4\frac{A}{2}},\ y'=\frac{1}{\sin^4\frac{B}{2}},\ z'=\frac{1}{\sin^4\frac{C}{2}},\\ A'=\frac{\sin^3\frac{A}{2}+\sin^3\frac{B}{2}}{\sin^4\frac{A}{2}},\ B'=\frac{\sin^3\frac{B}{2}+\sin^3\frac{C}{2}}{\sin^4\frac{B}{2}},\ C'=\frac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin^4\frac{C}{2}}\end{array}\right)\overset{\text{via }(1)}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{\sin^3\frac{A}{2}+\sin^3\frac{B}{2}}{\sin^4\frac{A}{2}}\cdot\frac{\sin^3\frac{B}{2}+\sin^3\frac{C}{2}}{\sin^4\frac{B}{2}}\right)}\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{\prod_{cyc}\left(\frac{\sin^3\frac{A}{2}+\sin^3\frac{B}{2}}{\sin^4\frac{A}{2}}\right)^2}}\overset{Cesaro}{\geq}$$

$$3.\sqrt[3]{\frac{8}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}\geq 3.\sqrt[3]{\frac{8}{\frac{1}{8}}}=12\ \ \forall\,\Delta\, ABC,''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$