ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\frac{m_a+2m_b}{m_a+2m_c}\right)^3+\left(\frac{m_b+2m_c}{m_b+2m_a}\right)^3+\left(\frac{m_c+2m_a}{m_c+2m_b}\right)^3\geq\frac{24r^3}{9R^3-64r^3}$$

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$Lemma: \forall\, a,b,c>0$

$$\frac{a+2b}{a+2c}+\frac{b+2c}{b+2a}+\frac{c+2a}{c+2b}\geq 3$$

$Proof: We\ need\ to\ show:$

$$\frac{a+2b}{a+2c}+\frac{b+2c}{b+2a}+\frac{c+2a}{c+2b}\geq 3$$

$$\left(\frac{a+2b}{a+2c}+1\right)+\left(\frac{b+2c}{b+2a}+1\right)+\left(\frac{c+2a}{c+2b}+1\right)\geq 6$$

$$\frac{2(a+b+c)}{a+2c}+\frac{2(a+b+c)}{b+2a}+\frac{2(a+b+c)}{c+2b}\geq 6$$

$$(a+b+c)\left(\frac{1}{a+2c}+\frac{1}{b+2a}+\frac{1}{c+2b}\right)\geq 3\ (1)$$

$$Let: a+2c=x, b+2a=y, c+2b=z\ then\ x+y+z=3(a+b+c)$$

$$(a+b+c)=\frac{1}{3}(x+y+z)$$

$$Now\ using\ above\ result\ from\ (1) we\ need\ to\ show:$$

$$\frac{1}{3}(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 3\ or, (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9$$

$$This\ is\ true\ by\ Cauchy\ Schwarz\ inequality:$$

$$\left(\frac{m_a+2m_b}{m_a+2m_c}\right)^3+\left(\frac{m_b+2m_c}{m_b+2m_a}\right)^3+\left(\frac{m_c+2m_a}{m_c+2m_b}\right)^3\overset{CBS}{\geq}$$

$$\geq\frac{1}{9}\left(\frac{m_a+2m_b}{m_a+2m_c}+\frac{m_b+2m_c}{m_b+2m_a}+\frac{m_c+2m_a}{m_c+2m_b}\right)^3\overset{Lemma}{\geq}$$

$$\geq\frac{1}{9}(3)^3=3=\frac{3R^3}{R^3}=\frac{3R^3}{9R^3-8R^3}\overset{Euler}{\geq}\frac{3(2r)^3}{9R^3-8(2r)^3}\geq\frac{24r^3}{9R^3-64r^3}$$

$$Equality\ holds\ for\ an\ equilateral\ triangle.$$