ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\right)}+\cot\frac{B}{2}.\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}}}{\cot\frac{A}{2}.\sqrt{\cot\frac{B}{2}}+\cot\frac{C}{2}.\sqrt{\cot\frac{A}{2}}}\geq 3\sqrt[4]{12}$$

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$$\forall\, A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\geq\sqrt{3\sum_{cyc}A'B'}$$

$$(\text{via Walter Janous})\to 1$$

$$\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\right)}+\cot\frac{B}{2}.\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}}}{\cot\frac{A}{2}.\sqrt{\cot\frac{B}{2}}+\cot\frac{C}{2}.\sqrt{\cot\frac{A}{2}}}$$

$$=\frac{\cot\frac{B}{2}\sqrt{\cot\frac{C}{2}}.\left(\sqrt{\frac{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}}{\cot\frac{B}{2}}}+\sqrt{\frac{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}}{\cot\frac{C}{2}}}\right)}{\cot\frac{C}{2}.\sqrt{\cot\frac{A}{2}}+\cot\frac{A}{2}.\sqrt{\cot\frac{B}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\cot\frac{B}{2}\sqrt{\cot\frac{C}{2}},\, y'=\cot\frac{C}{2}.\sqrt{\cot\frac{A}{2}},\, z'=\cot\frac{A}{2}.\sqrt{\cot\frac{B}{2}},\\ A'=\sqrt{\frac{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}}{\cot\frac{A}{2}}},\, B'=\sqrt{\frac{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}}{\cot\frac{B}{2}}},\, C'=\sqrt{\frac{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}}{\cot\frac{C}{2}}}\end{array}\right)$$

$$\overset{\text{via } 1}{\geq}\sqrt{3\sum_{cyc}\left(\sqrt{\frac{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}}{\cot\frac{A}{2}}}.\sqrt{\frac{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}}{\cot\frac{B}{2}}}\right)}\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{\prod_{cyc}\left(\frac{\cot^2\frac{A}{2}+\cot^2\frac{B}{2}}{\cot\frac{A}{2}}\right)}}\overset{Cesaro}{\geq}$$

$$3.\sqrt[6]{\frac{8\prod_{cyc}\cot^2\frac{A}{2}}{\prod_{cyc}\cot\frac{A}{2}}}=3.\sqrt[6]{\frac{8s^3}{rs^2}}=3.\sqrt[6]{\frac{8s}{r}}\overset{Mitrinovic}{\geq}3.\sqrt[6]{8.3\sqrt{3}}=3.\sqrt{2\sqrt{3}}=3.\sqrt{\sqrt{12}}$$

$$\therefore \sum_{\text{cyc}} \frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}\left(\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\right)}+\cot\frac{B}{2}.\sqrt{\cot^2\frac{C}{2}+\cot^2\frac{A}{2}}}{\cot\frac{A}{2}.\sqrt{\cot\frac{B}{2}}+\cot\frac{C}{2}.\sqrt{\cot\frac{A}{2}}} \geq 3.\sqrt[4]{12} \ \ \forall \ \Delta \ ABC,$$

$$'' ='' \text{ iff } \Delta \ ABC \text{ is equilateral (QED)}$$