# ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC the following relationship holds :**

$$\sum_{cyc}\frac{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{A}{2}+\tan\frac{B}{2}}+\pi^{2\tan\frac{C}{2}}.e^{\tan\frac{A}{2}+\tan\frac{C}{2}}}{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{B}{2}}+\pi^{\tan\frac{B}{2}+\tan\frac{C}{2}}.e^{\tan\frac{C}{2}}}\ge 3e^{\frac{\sqrt{3}}{3}}$$

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$$\forall\, A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\ge\sqrt{3\sum_{cyc}A'B'}$$

(via Walter Janous) $\to$ (1) and, $\displaystyle\sum_{cyc}\frac{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{A}{2}+\tan\frac{B}{2}}+\pi^{2\tan\frac{C}{2}}.e^{\tan\frac{A}{2}+\tan\frac{C}{2}}}{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{B}{2}}+\pi^{\tan\frac{B}{2}+\tan\frac{C}{2}}.e^{\tan\frac{C}{2}}}$

$$=\sum_{cyc}\frac{xyXY+z^2XZ}{xyY+yzZ}\left(\begin{matrix}x=\pi^{\tan\frac{A}{2}},y=\pi^{\tan\frac{B}{2}},z=\pi^{\tan\frac{C}{2}},\\ X=e^{\tan\frac{A}{2}},Y=e^{\tan\frac{B}{2}},Z=e^{\tan\frac{C}{2}}\end{matrix}\right)=\sum_{cyc}\frac{\frac{XY}{z}+\frac{zXZ}{xy}}{\frac{Y}{z}+\frac{Z}{x}}$$

$$=\sum_{cyc}\frac{\frac{X}{y}\left(\frac{Yy}{z}+\frac{Zz}{x}\right)}{\frac{Y}{z}+\frac{Z}{x}}=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{X}{y},y'=\frac{Y}{z},z'=\frac{Z}{x},A'=\frac{Xx}{y},B'=\frac{Yy}{z},C'=\frac{Zz}{x}\right)\overset{\text{via (1)}}{\ge}\sqrt{3\sum_{cyc}\left(\frac{Xx}{y}.\frac{Yy}{z}\right)}$$

$$\overset{AM-GM}{\ge}3.\sqrt{\sqrt[3]{X^2Y^2Z^2}}=3.\sqrt[3]{e^{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}}=3.\sqrt[3]{e^{\frac{4R+r}{s}}}\overset{\text{Doucet (or Trucht)}}{\ge}3.\sqrt[3]{e^{\sqrt{3}}}$$

$$=3e^{\frac{\sqrt{3}}{3}}\therefore\sum_{cyc}\frac{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{A}{2}+\tan\frac{B}{2}}+\pi^{2\tan\frac{C}{2}}.e^{\tan\frac{A}{2}+\tan\frac{C}{2}}}{\pi^{\tan\frac{A}{2}+\tan\frac{B}{2}}.e^{\tan\frac{B}{2}}+\pi^{\tan\frac{B}{2}+\tan\frac{C}{2}}.e^{\tan\frac{C}{2}}}\ge 3e^{\frac{\sqrt{3}}{3}}\ \forall\,\Delta\,ABC,$$

$''=''$ iff $\Delta$ ABC is equilateral (QED)