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In any ΔABC the following relationship holds :

$$\frac{\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2}}{\sin \frac{A}{2} \sin \frac{B}{2} + \sin \frac{C}{2}} + \frac{\sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2} + \sin \frac{A}{2}} + \frac{\sin^2 \frac{C}{2} + \sin^2 \frac{A}{2}}{\sin \frac{C}{2} \sin \frac{A}{2} + \sin \frac{B}{2}} \geq 2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{LHS} &\stackrel{\text{AM-GM}}{\geq} 2 \sum_{\text{cyc}} \frac{xy}{xy+z} \left(x = \sin \frac{A}{2}, y = \sin \frac{B}{2}, z = \sin \frac{C}{2} \right) \stackrel{?}{=} 2 \\ &\Leftrightarrow \sum_{\text{cyc}} (xy(yz+x)(zx+y)) \stackrel{?}{=} \prod_{\text{cyc}} (xy+z) \\ &\Leftrightarrow 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + \sum_{\text{cyc}} \sin^2 \frac{A}{2} - 1 \stackrel{?}{=} 0 \Leftrightarrow \frac{r}{2R} - 1 + \frac{2R-r}{2R} \stackrel{?}{=} 0 \rightarrow \text{true} \end{aligned}$$

$$\therefore \frac{\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2}}{\sin \frac{A}{2} \sin \frac{B}{2} + \sin \frac{C}{2}} + \frac{\sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2} + \sin \frac{A}{2}} + \frac{\sin^2 \frac{C}{2} + \sin^2 \frac{A}{2}}{\sin \frac{C}{2} \sin \frac{A}{2} + \sin \frac{B}{2}} \geq 2 \quad \forall \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)