ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC$ the following relationship holds:

$$\sqrt{m_a^2+m_am_b+m_b^2}+\sqrt{m_c^2+m_cm_b+m_b^2}+\sqrt{m_c^2+m_am_c+m_a^2}\ge 9\sqrt{3}r$$

*Proposed by Zaza Mzhavanadze-Georgia*

*Solution by Tapas Das-India*

$$m_am_bm_c\ge h_ah_bh_c\overset{GM\text{-}HM}{\ge}\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^3=(3r)^3=27r^3\ (1)$$

$$\sqrt{m_a^2+m_am_b+m_b^2}+\sqrt{m_c^2+m_cm_b+m_b^2}+\sqrt{m_c^2+m_am_c+m_a^2}=$$

$$=\sum\sqrt{m_a^2+m_am_b+m_b^2}\overset{\grave{A}M-GM}{\ge}\sum\sqrt{2m_am_b+m_am_b}=$$

$$=\sum\sqrt{3m_am_b}\overset{\grave{A}M-GM}{\ge}3\sqrt[3]{3\sqrt{3}\times m_am_bm_c}\overset{(1)}{\ge}3\sqrt[3]{3\sqrt{3}\times 27r^3}=9\sqrt{3}r$$

Equality holds for an equilateral triangle.