ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a^2+b^2}{ab+c}+\frac{b^2+c^2}{bc+a}+\frac{c^2+a^2}{ac+b}\geq\frac{12\sqrt{3}r}{\sqrt{3}R+1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$ab+bc+ca+a+b+c=s^2+r^2+4Rr+2s=$$

$$=s^2+r(4R+r)+2s\overset{s^2\geq 3r(4R+r)}{\leq}s^2+\frac{s^2}{3}+2s=\frac{4s^2}{3}+2s\ (1)$$

$$\frac{a^2+b^2}{ab+c}+\frac{b^2+c^2}{bc+a}+\frac{c^2+a^2}{ac+b}=\sum\frac{a^2+b^2}{ab+c}\overset{CBS}{\geq}\sum\frac{\frac{(a+b)^2}{2}}{ab+c}=$$

$$=\frac{1}{2}\sum\frac{(a+b)^2}{ab+c}\overset{Bergstrom}{\geq}\frac{1}{2}\cdot\frac{(2a+2b+2c)^2}{ab+bc+ca+a+b+c}=$$

$$=\frac{1}{2}\cdot\frac{4(a+b+c)^2}{ab+bc+ca+a+b+c}\overset{(1)}{\geq}\frac{2\times 4s^2}{\frac{4s^2}{3}+2s}=\frac{4s}{\frac{2s}{3}+1}\overset{Mirinovic}{\geq}\frac{4\times 3\sqrt{3}r}{\frac{2}{3}\cdot\frac{3\sqrt{3}R}{2}+1}=\frac{12\sqrt{3}r}{\sqrt{3}R+1}$$

Equality holds for an equilateral triangle.